The Automorphism Group of the Vertex Operator Algebra V_L^+ for an even lattice L without roots

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Abstract

The automorphism group of the vertex operator algebra V_L^+ is studied by using its action on isomorphism classes of irreducible V_L^+-modules. In particular, the shape of the automorphism group of V_L^+ is determined when L is isomorphic to an even unimodular lattice without roots, $\sqrt{2}R$ for an irreducible root lattice R of type ADE and the Barnes-Wall lattice of rank 16.

Introduction

Since the introduction of vertex operator algebras (VOAs) [Bo, FLM], it has been a basic problem to determine the automorphism group of a VOA. For example, as is well known, the automorphism group of the moonshine module is the Monster simple group (cf. [FLM]). We believe that there are many VOAs having an "interesting" (finite) group as the automorphism group.

The main objective of this paper is the VOA V_L^+ which is the fixed-point subspace of the lattice VOA V_L associated with a (positive-definite) even lattice L with respect to an automorphism θ_{V_L} induced from the (-1) isometry of L. In this paper, we study the automorphism group $\mathrm{Aut}(V_L^+)$ of V_L^+. Clearly $\mathrm{Aut}(V_L^+)$ has the subgroup H_L obtained as the restriction of the centralizer of θ_{V_L} in $\mathrm{Aut}(V_L)$. It was proved in [DG] that $\mathrm{Aut}(V_L^+)$ coincides with H_L when L is isomorphic to an even lattice L of rank one except the lattice $2A_1$. On the other hand, it was shown in [DG, Gr2] that $\mathrm{Aut}(V_L^+)$ does not coincide with H_L when L is isomorphic to $2A_1$ or $\sqrt{2}E_8$. We call an automorphism of V_L^+ not belonging to H_L an extra automorphism. So the natural question occurs: "When does V_L^+ have extra automorphisms?".

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Our main result is to give a uniform method of determining $\mathrm{Aut}(V_L^+)$ for an even lattice L without roots (Section 3.4). As a corollary, we will answer the question above. More precisely, for an even lattice L without roots, we will show that V_L^+ has extra automorphisms if and only if L is obtained from a binary code by Construction B (Proposition 3.16 (2)). Hence we obtain a new relation between the VOA V_L^+ and the lattice L. As an application, we will determine $\mathrm{Aut}(V_L^+)$ when L is isomorphic to an even unimodular lattice without roots, $\sqrt{2}R$ for an irreducible root lattice R and the Barnes-Wall lattice of rank 16 (Theorem 4.1, 4.8 and Table 2).

Let us explain our basic idea. Given a group G, one of effective methods of determining the group structure of G is to find a set X on which G acts as permutations and to determine both the kernel and image of the natural group homomorphism from G to the symmetric group $Sym(X)$ on X. In our case, the group G is $\mathrm{Aut}(V_L^+)$, and we take, as such an X, the set S_L of all isomorphism classes of irreducible V_L^+-modules.

We now recall the action of automorphisms of a VOA on isomorphism classes of its irreducible modules form [DM1]. Let (M, Y_M) be a module for a VOA V. For an automorphism g of V, we set $Y_M^g(v,z) = Y_M(gv,z)$ $(v \in V)$. Then (M, Y_M^g) is a V-module. If M is irreducible then so is M^g, hence automorphisms of V acts on the set of all isomorphism classes of irreducible V-modules.

We obtained the set S_L on which $\mathrm{Aut}(V_L^+)$ acts. However, $Sym(S_L)$ is too large to describe $\mathrm{Aut}(V_L^+)$ directly. So we replace S_L with a subset P of S_L preserved by the action of $\mathrm{Aut}(V_L^+)$, and consider a structure on P associated with the fusion rules. Since the action of $\mathrm{Aut}(V_L^+)$ preserves the fusion rules, we obtain a group homomorphism from $\mathrm{Aut}(V_L^+)$ to the group consisting of all elements of $Sym(P)$ preserving the structure of P. This homomorphism gives us sufficient information to determine $\mathrm{Aut}(V_L^+)$.

Precisely, we proceed as follows:

(1) Describe the stabilizer H_L of the isomorphism class of V_L^- in $\mathrm{Aut}(V_L^+)$.

(2) Determine the orbit Q_L of the isomorphism class of V_L^-.

(3) Find a subset P of S_L such that P contains the orbit Q_L and forms a vector space under the fusion rules.

Here V_L^- is the irreducible V_L^+-module which is the (-1)-eigenspace of the involution θ_{V_L}.

By performing the steps above, we can determine $\mathrm{Aut}(V_L^+)$ in the following way. By the step (3), we obtain a natural group homomorphism $\zeta_{V_L^+}$ from $\mathrm{Aut}(V_L^+)$ to the general linear group on the vector space P. So it suffices to determine the kernel and image of $\zeta_{V_L^+}$. Since the kernel is a subgroup of the stabilizer H_L, we can determine it by step (1). Clearly $\mathrm{Aut}(V_L^+)$ acts transitively on the orbit Q_L. Hence the index of the stabilizer H_L in $\mathrm{Aut}(V_L^+)$ is equal to the cardinality of the orbit Q_L, and so is the index of the image of the stabilizer H_L in the image of $\mathrm{Aut}(V_L^+)$ under the homomorphism $\zeta_{V_L^+}$. By the step (1), we can determine the image of the stabilizer H_L. In this stage, we can use group theoretical facts on general linear groups, and one can compute the image of $\zeta_{V_L^+}$ in principle.

Let us explain our arguments in the steps (1), (2) and (3) in detail. First we describe the stabilizer H_L of the isomorphism class $[0]^-$ of V_L^-. Clearly, any automorphism of V_L that commutes with the involution θ_{V_L} preserves both V_L^+ and V_L^-. Hence by restriction, we obtain a group homomorphism from the centralizer $C_{\mathrm{Aut}(V_L)}(\theta_{V_L})$ of θ_{V_L} in $\mathrm{Aut}(V_L)$ to H_L. We will show that this homomorphism is surjective, namely we will prove that for any element of H_L, there exists its preimage in $C_{\mathrm{Aut}(V_L)}(\theta_{V_L})$ (Theorem 3.3 and Proposition 3.10). By using the description of $\mathrm{Aut}(V_L)$ given in [DN1], we can compute the stabilizer H_L of $[0]^-$ in principle and we complete the step (1).

Next we study the orbit Q_L of the isomorphism class $[0]^-$ of V_L^-. When the lattice L does not have roots, then the degree one subspace of V_L^+ is zero and the automorphism group of V_L^+ is expected to be finite. (This is indeed the case as we will show in Proposition 3.11.) We are interested in such a case, so we assume that L has no roots. If the isomorphism class W of an irreducible V_L^+-module M belongs to Q_L, then the following must be satisfied:

(a) the dimension of the degree 1 subspace of M is equal to that of V_L^-,

(b) the fusion rule $W \times W = V_L^+$ holds.

We note that the isomorphism class of any irreducible V_L^+-module is either of untwisted type or of twisted type, which are came from (untwisted) V_L-modules or θ_{V_L}-twisted V_L-modules respectively (cf. [AD]). For simplicity, we consider the following two cases:

(I) Q_L does not contain isomorphism classes of irreducible V_L^+-modules of twisted type,

(II) Q_L contains isomorphism classes of irreducible V_L^+-modules of twisted type.

In the case (I), let W be the isomorphism class of an irreducible V_L^+-module of untwisted type satisfying the conditions (a) and (b), and suppose that W is not the isomorphism class $[0]^-$ of V_L^-. We will prove that the lattice L comes from the Construction B, a well-known procedure which makes a lattice from a binary code, and that W and $[0]^-$ are actually conjugate each other, that is, there exists an element of $\mathrm{Aut}(V_L^+)$ which permutes W and $[0]^-$. For an even lattice N obtained by Construction B, it was shown in [FLM] that there exists an automorphism of V_N^+ which cannot lift to an automorphism of V_N. In our case, the automorphism of V_L^+ permuting W and $[0]^-$ is given by that described as in [FLM]. By these consideration, we determine Q_L and complete the step (2) (Theorem 3.15). Then, by using the fusion rules given in [ADL], we show that the union of Q_L and the isomorphism class of V_L^+ forms a vector space over $\mathbb{F}_2$ under the fusion rules (Proposition 3.17), and this completes the step (3).

In the case (II), let W be an element of Q_L of twisted type. By the assumption, the graded dimensions of V_L^- and W are the same, and the fusion rules of V_L^- and those of W are similar. In this case, by the results in [ADL, FLM], we deduce that L is a 2-elementary totally even lattice of rank 8 or 16 (Proposition 3.14). This implies that L is isomorphic to $\sqrt{2}E_8$ or the Barnes-Wall lattice Λ_{16} of rank 16 under the assumption that L has no roots (cf. [CS, Qu]). By using the fusion rules of V_L^+ given in [ADL], if L is 2-elementary

totally even then the set S_L of all isomorphism classes of irreducible V_L^+-modules forms a vector space over $\mathbb{F}_2$ under the fusion rules. Furthermore, when the rank of L is a multiple of 8, we define a natural $\mathrm{Aut}(V_L^+)$-invariant quadratic form on this vector space (Theorem 3.8), and obtain a group homomorphism from $\mathrm{Aut}(V_L^+)$ to an orthogonal group. By using this homomorphism, we determine $\mathrm{Aut}(V_L^+)$ when L is isomorphic to $\sqrt{2}E_8$ and Λ_{16}. As a corollary, we see that these lattices satisfy the condition (II).

The organization of this paper is as follows: In Section 1, we recall some definitions and facts necessary in this paper. In Section 2, we review the properties of the VOA V_L^+, such as the irreducible modules, the fusion rules, the graded dimensions and automorphisms. In Section 3, we study the action of $\mathrm{Aut}(V_L^+)$ on the set of all isomorphism classes of irreducible V_L^+-modules, and give a method of determining the automorphism group of V_L^+ for an even lattice L without roots. In Section 4, applying this method to many important lattices L, we determine $\mathrm{Aut}(V_L^+)$.

Throughout this paper, we will work over the field $\mathbb{C}$ of complex numbers unless otherwise stated. We denote by $\mathbb{Z}_q$ the set of integers by $\mathbb{Z}$ and the ring of integers modulo q . We often identify $\mathbb{Z}_2$ with the field $\mathbb{F}_2$ of two elements. We denote the elementary abelian 2-group of order 2^n simply by 2^n and the root lattice of type R simply by R. We use the notation of [ATLAS] for classical groups except that we denote by $O_n^+(2)$ the orthogonal group of degree n over $\mathbb{F}_2$ of plus type and by $\Omega_n^+(2)$ its simple normal subgroup of index 2 . We also use the notation of [ATLAS] for group extensions: For groups A and B, $A.B$ denotes any group extension for which the quotient by a normal subgroup isomorphic to A is isomorphic to B, $A:B$ indicates any case of $A.B$ for which the extension is split and $A \cdot B$ indicates any case of $A.B$ for which the extension is not split.

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1 Preliminaries

In this section, we recall or give some definitions and facts necessary in this paper.

1.1 Fusion rules, graded dimensions and automorphisms

In this subsection, we recall the fusion rules and the graded dimensions from [FLM, FHL] and the action of automorphisms of a VOA on its modules from [DM1]. For details of the definition of VOAs, see [Bo, FLM].

Let V be a VOA and let M^1, M^2, M^3 be V-modules. Let $I_V \binom{M^3}{M^1\ M^2}$ denote the vector space generated by all intertwining operators of type $\binom{M^3}{M^1\ M^2}$ (for the definition see [FHL]). We set

$$N_{M^1 M^2}^{M^3} = \dim I_V \begin{pmatrix} M^3 \\ M^1\ M^2 \end{pmatrix}.$$

This number is called the *fusion rule*. The following symmetry on the fusion rules is well known.

Proposition 1.1. [FHL] *Let M^1, M^2, M^3 be V-modules. Then*

$$N_{M^1 M^2}^{M^3} = N_{M^2 M^1}^{M^3}.$$

Let $\{W^i\}_{i \in I}$ be the set of all isomorphism classes of irreducible V-modules and let M^i be a representative of W^i for $i \in I$. We denote the fusion rules by using the following formal product:

$$W^i \times W^j = \sum_{k \in I} N_{M^i M^j}^{M^k} W^k. \tag{1.1}$$

We note that this product is independent of the choice of representatives, and that $W^i \times W^j = W^j \times W^i$ by Proposition 1.1. Later, when V is the VOA V_L^+, we will find a subset I' of I such that for any $i, j \in I'$ there exists a unique element $k \in I'$ satisfying $W^i \times W^j = W^k$, and will view this formal product as a binary operation on $\{W^i\}_{i \in I'}$. We often denote the isomorphism class of an irreducible V-module M simply by M.

Let $B = \oplus_{i \in \mathbb{C}} B_i$ be a graded vector space over $\mathbb{C}$ satisfying $\dim B_i < \infty$ for all $i \in \mathbb{C}$. Then the *graded dimension* of B is the formal sum

$$\dim_* B = \sum_{i \in \mathbb{C}} (\dim B_i) q^i.$$

Since any module of a VOA V is a graded vector space whose homogeneous subspaces are finite dimensional, we may consider its graded dimension. Clearly, if V-modules M^1 and M^2 are isomorphic then $\dim_* M^1 = \dim_* M^2$. Hence for the isomorphism class W of a V-module M, we often denote by $\dim_* W$ the graded dimension of M.

Note 1.2. The graded dimension is often called the character. However, since we consider characters of groups later, we use the term graded dimension in order to avoid confusion.

An *automorphism* of a VOA $(V, Y, \mathbf{1}, \omega)$ is a linear isomorphism $g : V \to V$ satisfying $Y(gv, z)g = gY(v, z)$ for all $v \in V$ that fixes the Virasoro element ω. Let $\mathrm{Aut}(V)$ denote the group of all automorphisms of V. It is easy to see that any automorphism preserves the grading and fixes the vacuum vector $\mathbf{1}$.

Now we recall the action of automorphisms of V on its modules. Let (M, Y_M) be a V-module. For an automorphism g of V, the linear map Y_{M^g} is defined by

$$Y_{M^g}(v, z) = Y_M(gv, z).$$

Then $M^g = (M, Y_{M^g})$ is a V-module. The following proposition concerning M^g is an easy exercise.

Proposition 1.3. (1) *If M is irreducible then M^g is also irreducible. In particular* $\mathrm{Aut}(V)$ *acts on the set of all isomorphism classes of irreducible V-modules.*

(2) *The graded dimensions of M and M^g are the same.*

(3) *Suppose $V_0 = \mathbb{C}\mathbf{1}$. Then, for $v \in V_1$, $g = \exp(v_0)$ is an automorphism of V and $M^g \cong M$ as V-modules.*

(4) *Let M^1, M^2, M^3 be irreducible V-modules. Then $N_{M^1,M^2}^{M^3} = N_{M^{1,g},M^{2,g}}^{M^{3,g}}$.*

1.2 Construction B and 2-elementary totally even lattices

In this subsection, we study lattices obtained by Construction B from binary codes and 2-elementary totally even lattices. For details of the definitions of lattices and codes, see [CS].

Let n be a positive integer and set $\Omega_n = \{1, 2, \ldots, n\}$. Let $\{\alpha_i |\ i \in \Omega_n\}$ be an orthogonal basis of $\mathbb{R}^n$ satisfying $\langle \alpha_i, \alpha_j\rangle = 2\delta_{i,j}$. For a subset $J \subset \Omega_n$, we set $\alpha_J = \sum_{i\in J} \alpha_i$. Let C be a (binary linear) code of length n, which we regard as a subset of the power set of Ω_n. Then the lattice

$$L_B(C) = \sum_{c\in C} \mathbb{Z}\frac{1}{2}\alpha_c + \sum_{i,j\in\Omega_n} \mathbb{Z}(\alpha_i + \alpha_j)$$

is called *the lattice obtained by Construction B from C* associated with $\{\alpha_i |\ i \in \Omega_n\}$. The following propositions concerning $L_B(C)$ are well-known.

Proposition 1.4. *Let $\tilde{C}$ be a set of generators of C. Then $L_B(C)$ is generated by $\{\alpha_i \pm \alpha_j,\ \frac{1}{2}\alpha_c |\ i, j \in \Omega_n,\ c \in \tilde{C}\}$ as a $\mathbb{Z}$-module.*

Proposition 1.5. [CS] *Let C be a doubly even code of length n.*

(1) *The lattice $L_B(C)$ is even and its determinant is $2^{n-2\dim C+2}$.*

(2) *The dual lattice $L_B(C)^\circ$ of $L_B(C)$ is given by*

$$L_B(C)^\circ = L_B(C^\perp) + \mathbb{Z}\alpha_1 + \mathbb{Z}\frac{\alpha_{\Omega_n}}{4}.$$

(3) *If the minimum weight of C is greater than 4 then the minimum norm of $L_B(C)$ is 4, and in particular $L_B(C)$ has no roots.*

The following are typical examples of lattices obtained by Construction B.

Example 1.6. [CS]

Table 1: Automorphism groups of lattices

R	$\det R$	$\mathrm{Aut}(R)$
A_1	2	$\mathbb{Z}_2$
A_n $(n \neq 1)$	$n+1$	$Sym_{n+1} \times 2$
D_4	4	$(2^3 : Sym_4) : Sym_3$
D_n $(n \geq 5)$	4	$2^{n-1} : Sym_n : 2$
E_6	3	$2.U_4(2) : 2$
E_7	2	$2.Sp_6(2)$
E_8	1	$2.O_8^+(2)$
Λ_{16}	2^8	$2_+^{1+8} \cdot \Omega_8^+(2)$

(1) Let C be the code of length n consisting of the all-zero codeword. Then $L_B(C)$ is isomorphic to $\sqrt{2}D_n$, where D_1, D_2 and D_3 are regarded as $2A_1$, $A_1 \oplus A_1$ and A_3 respectively.

(2) Let C be the code of length 8 consisting of the all-zero and all-one codewords. Then $L_B(C)$ is isomorphic to $\sqrt{2}E_8$.

(3) Let $RM(4,1)$ denote the first order Reed-Muller code of length 2^4. Then $L_B(RM(4,1))$ is isomorphic to the Barnes-Wall lattice of rank 16.

Remark 1.7. The automorphism groups and the determinants of the irreducible root lattices and the Barnes-Wall lattice Λ_{16} of rank 16 are summarized in Table 1 (cf. Chapter 4 of [CS]).

The following proposition will be used in Section 3.3. For a subset U of $\mathbb{R}^n$ and $i \in \mathbb{Z}_{>0}$, we denote $U_i = \{v \in U \mid \langle v, v\rangle = i\}$.

Proposition 1.8. *Let L be an even lattice of rank n without roots. If there exists a vector $\lambda \in L^\circ \cap (L/2)$ such that $|(\lambda+L)_2| = 2n$ then L is obtained by Construction B associated with an orthogonal basis of $\mathbb{R}\otimes_{\mathbb{Z}} L$ consisting of vectors in $(\lambda+L)_2$.*

Proof. First, let us show that $(\lambda+L)_2$ forms a root system of type A_1^n. Since $\langle\lambda,\lambda\rangle \in 2\mathbb{Z}$ and $\lambda \in L^\circ$, the lattice $N = L + \mathbb{Z}\lambda = L \cup (\lambda + L)$ is even. It follows from $L_2 = \phi$ that $(\lambda+L)_2$ forms a root system. Assume that it is not of type A_1^n. Since $|(\lambda+L)_2|$ is equal to the number of roots of A_1^n, the set $(\lambda+L)_2$ contains a root system not of type A_1. Hence there exist $x, y \in (\lambda+L)_2$ such that $\langle x, y\rangle = -1$. Then the norm of $x+y$ is 2 and $x+y \in L$, which is a contradiction for $L_2 = \phi$. Thus $(\lambda+L)_2$ forms a root system of type A_1^n. In particular $(\lambda+L)_2$ contains an orthogonal basis of $\mathbb{R}\otimes_{\mathbb{Z}} L$.

Next we show that L is obtained by Construction B. Let $\{\alpha_i \mid i \in \Omega_n\}$ be a basis of $\mathbb{R}\otimes L$ consisting of vectors in $(\lambda+L)_2$ and set $M = \oplus_{i\in\Omega_n}\mathbb{Z}\alpha_i$. Then $M^\circ = M/2$ and $M \subset N \subset M^\circ$. Let C be the code obtained as the image of N/M under the

homomorphism of vector spaces over $\mathbb{F}_2$: $M^\circ/M \to \mathbb{Z}_2^n$, $\sum_{i\in\Omega_n} c_i\alpha_i/2 \mapsto (\bar{c}_1, \bar{c}_2, \dots, \bar{c}_n)$. Then we have $N = \sum_{i\in\Omega_n}\mathbb{Z}\alpha_i + \sum_{c\in C}\mathbb{Z}\alpha_c/2$. It follows from $\{\alpha_i|\ i\in\Omega_n\}\subset \lambda + L$ and $2\lambda\in L$ that $\alpha_i\pm\alpha_j\in L$. Let $\{c_1, c_2, \dots, c_k\}$ be a basis of C. Then there exists $D\subset\Omega_n$ such that $\alpha_{c_i}/2$ belongs to $\epsilon_D(L)$ for all i, where ϵ_D denotes the linear map of $\mathbb{R}^n$ given by $\epsilon_D(\alpha_i) = \alpha_i$ if $i\in D$, and $\epsilon_D(\alpha_i) = -\alpha_i$ if $i\notin D$. Since L contains $\alpha_i\pm\alpha_j$ for all $i,j\in\Omega_n$, so does $\epsilon_D(L)$. Hence it follows from Proposition 1.4 that $\epsilon_D(L)\supset L_B(C)$. Since $|N/L_B(C)| = |N/L| = 2$, we have $\epsilon_D(L) = L_B(C)$. Thus L is obtained by Construction B from C associated with $\{\epsilon_D(\alpha_i)|\ i\in\Omega_n\}$. □

In the rest of this section, we consider 2-elementary totally even lattices. The VOA V_L^+ for such a lattice will be studied in Section 3.2

An even lattice L is called 2-*elementary totally even* if $2L^\circ\subset L$ and if $\sqrt{2}L^\circ$ is even. Then the lattice $\sqrt{2}E_8$ and the Barnes-Wall lattice of rank 16 are characterized as follows:

Proposition 1.9. (1) (cf. [CS]) *The lattice $\sqrt{2}E_8$ is the unique 2-elementary totally even lattice of rank 8 with determinant 2^8 up to isomorphism.*

(2) [Qu, Theorem 4] *The Barnes-Wall lattice of rank 16 is the unique 2-elementary totally even lattice of rank 16 with determinant 2^8 without roots up to isomorphism.*

2 Vertex operator algebra V_L^+

In this section, we review the VOA V_L^+ and its automorphisms.

2.1 V_L^+ and its irreducible modules

In this subsection, we review the VOA V_L^+ and its irreducible modules as well as their properties from [FLM, DN2, AD, Ab, ADL].

Let L be an even lattice of rank n equipped with a positive-definite symmetric $\mathbb{Z}$-bilinear form $\langle\cdot,\cdot\rangle$. We set $\mathfrak{h}_L = \mathbb{C}\otimes_{\mathbb{Z}} L$ and regard L as a subgroup of $\mathfrak{h}_L$. We extend the form $\langle\cdot,\cdot\rangle$ to a $\mathbb{C}$-bilinear form on $\mathfrak{h}_L$. Let $L^\circ = \{v\in\mathfrak{h}_L|\ \langle v, L\rangle\subset\mathbb{Z}\}$ denote the dual lattice of L. For a subset $M\subset L^\circ$, we set $\mathfrak{h}_M = \mathbb{C}\langle M\rangle\subset\mathfrak{h}_L$.

Let q be the minimal positive even integer such that $\langle L^\circ, L^\circ\rangle\subset 2\mathbb{Z}/q$. By Remark 12.18 in [DL], there exists a central extension $\hat{L^\circ}$ of L° by the cyclic group $\langle\omega_q\rangle\cong\mathbb{Z}_q$

$$1\to\langle\omega_q\rangle\to\hat{L^\circ}\xrightarrow{-}L^\circ\to 0 \tag{2.1}$$

such that $[a,b] = \kappa_L^{\langle\bar{a},\bar{b}\rangle}$ if $\bar{a},\bar{b}\in L$, where $\kappa_L = \omega_q^{q/2}$. We often regard ω_q as a primitive q-th root of the unity. We may choose a $\mathbb{Z}$-bilinear 2-cocycle $\varepsilon(\cdot,\cdot): L^\circ\times L^\circ\to\mathbb{Z}_q$ such that $\varepsilon(\alpha,\alpha) = q\langle\alpha,\alpha\rangle/4$ and denote the corresponding section by $L^\circ\to\hat{L^\circ}$, $\alpha\mapsto e(\alpha)$. Let $\hat{L}$ be a central extension of L by $\langle\kappa_L\rangle\cong\mathbb{Z}_2$ such that $[a,b] = \kappa_L^{\langle\bar{a},\bar{b}\rangle}$ for $a,b\in\hat{L}$. We regard $\hat{L}$ as a subgroup of $\hat{L^\circ}$.

Let us consider the lattice VOA V_L. Form the induced $\hat{L^\circ}$-module $\mathbb{C}\{L^\circ\} = \mathbb{C}[\hat{L^\circ}]\otimes_{\mathbb{C}[\omega_q]}\mathbb{C}$, where $\mathbb{C}[\cdot]$ denotes the operation of forming the group algebra and ω_q acts on $\mathbb{C}$ by

the multiplication. For a subset $M \subset L^\circ$, we set $\hat{M} = \{a \in \hat{L^\circ} |\ \bar{a} \in M\}$, $\mathbb{C}\{M\} = \mathbb{C}[\hat{M}] \otimes_{\mathbb{C}[\omega_q]} \mathbb{C}$ and $V_M = S(\mathfrak{h}_M \otimes t^{-1}\mathbb{C}[t^{-1}]) \otimes \mathbb{C}\{M\}$. For convenience, we denote $h \otimes t^m$ by $h(m)$ for $h \in \mathfrak{h}_M$ and $m \in \mathbb{Z}$. It was shown in [Bo, FLM] that V_L has a simple VOA structure and $V_{\lambda+L}$ has an irreducible V_L-module structure for $\lambda + L \in L^\circ/L$.

The VOA V_L^+ is constructed as follows. Let θ_{L° be an automorphism of L° such that $\theta_{L^\circ}(\omega_q^m e(\alpha)) = \omega_q^m e(-\alpha)$, and let θ_{V_L} be the unique automorphism of the V_L-module V_{L° such that $\theta_{V_L}(h(m) \otimes 1) = -h(m) \otimes 1$ and $\theta_{V_L}(1 \otimes e(\alpha)) = 1 \otimes \theta_{L^\circ}(e(\alpha))$ for $h \in \mathfrak{h}_{L^\circ}$ and $\alpha \in L^\circ$. For a θ_{V_L}-stable subspace E, we set $E^\pm = \{v \in E |\ \theta_{V_L}(v) = \pm v\}$. Then V_L^+ is a subVOA of V_L, and $V_{\mu+L}$ and $V_{\lambda+L}^\pm$ are irreducible V_L^+-modules for $\mu \in L^\circ \setminus (L/2)$ and $\lambda \in L^\circ \cap (L/2)$. Such an irreducible V_L^+-module is said to be *of untwisted type*

Let us consider irreducible V_L^+-modules of twisted type. Set $K_L = \{a^{-1}\theta_{L^\circ}(a) |\ a \in \hat{L}\}$. Then K_L is a normal subgroup of $\hat{L}$. For an $\hat{L}/K_L$-module T on which κK_L acts by -1, we set $V_L^T = S(\mathfrak{h}_L \otimes t^{-1/2}\mathbb{C}[t^{-1}]) \otimes T$. It was shown in [FLM] that V_L^T has a V_L^+-module structure. Let ϕ be the unique commutative algebra automorphism of $S(\mathfrak{h}_L \otimes t^{-1/2}\mathbb{C}[t^{-1}])$ such that $\phi(h(m)) = -h(m)$ for $h \in \mathfrak{h}_L$ and $m \in 1/2 + \mathbb{Z}$, and let $\theta_{V_L^T}$ be the automorphism of the V_L^+-module V_L^T given by $\theta_{V_L^T}(x \otimes t) = \phi(x) \otimes t$. We also denote the ± 1-eigenspace of $\theta_{V_L^T}$ by $V_L^{T,\pm}$. Then $V_L^{T,\pm}$ are irreducible V_L^+-modules if T is irreducible. Such an irreducible V_L^+-module is said to be *of twisted type.*

We study irreducible $\hat{L}/K_L$-modules more precisely. For a subgroup G of $\hat{L}/K_L$ containing $\langle \kappa_L K_L \rangle$, we denote

$$X(G) = \{\chi \in \mathrm{Irr}(G) |\ \chi(\kappa_L K_L) = -1\},$$

where $\mathrm{Irr}(G)$ is the set of all irreducible characters of the group G. Applying Theorem 5.5.1 of [FLM] to $\hat{L}/K_L$, we obtain the following proposition.

Proposition 2.1. [FLM] *The restriction map $X(\hat{L}/K_L) \to X(Z(\hat{L}/K_L))$ is bijective, where $Z(\hat{L}/K_L)$ is the center of $\hat{L}/K_L$. Moreover the degree of any element of $X(\hat{L}/K_L)$ is equal to $|L/(L \cap 2L^\circ)|^{1/2}$.*

We denote the irreducible $\hat{L}/K_L$-module corresponding to $\chi \in X(Z(\hat{L}/K_L))$ by T_χ.

We have obtained some irreducible V_L^+-modules. In [DN2] (for rank one) and [AD] (for general rank), the irreducible V_L^+-modules were classified.

Theorem 2.2. [DN2, AD] *Let L be an even lattice. Then any irreducible V_L^+-module is isomorphic to one of $V_{\lambda+L}^\pm$ ($\lambda \in L^\circ \cap (L/2)$), $V_{\mu+L}$ ($\mu \in L^\circ \setminus (L/2)$) and $V_L^{T_\chi,\pm}$ ($\chi \in X(Z(\hat{L}/K_L))$).*

Remark 2.3. The irreducible V_L^+-modules in Theorem 2.2 are non-isomorphic except that $V_{\mu+L} \cong V_{-\mu+L}$ for $\mu \in L^\circ \setminus (L/2)$.

In this paper, we use the following notation of [ADL] for the isomorphism classes of irreducible V_L^+-modules: $[\lambda]^\pm$ ($\lambda \in L^\circ \cap (L/2)$), $[\mu]$ ($\mu \in L^\circ \setminus (L/2)$) and $[\chi]^\pm$ ($\chi \in X(Z(\hat{L}/K_L))$) denote the isomorphism classes of $V_{\lambda+L}^\pm$, $V_{\mu+L}$ and $V_L^{T_\chi,\pm}$ respectively

In [Ab] (for rank one) and [ADL] (for general rank), the fusion rules of V_L^+ were completely determined. In particular, we have the following proposition.

Proposition 2.4. [Ab, ADL]

(1) *Let $\lambda_1, \lambda_2 \in L^\circ \cap (L/2)$ and $\mu \in L^\circ \setminus (L/2)$. Let W be the isomorphism class of an irreducible V_L^+-module. Then*

$$\begin{aligned}
[0]^-\times\ \ [\lambda_1]^{\pm} &= [\lambda_1]^{\mp},\\
[0]^-\times\ \ [\mu] &= [\mu],\\
[\lambda_1]^+\times\ \ [\lambda_2]^+ &= [\lambda_1+\lambda_2]^{\varepsilon} \text{ for some } \varepsilon \in \{\pm\},\\
[0]^+\times\ \ W &= W,\\
[\mu]\times\ \ [\mu] &= [0]^+ + [0]^- + [2\mu].
\end{aligned}$$

(2) *Let M^1, M^2, M^3 be irreducible V_L^+-modules. Suppose that M^1 is of twisted type and that $N_{M^1,M^2}^{M^3} = 1$. If M^2 is of twisted type (resp. of untwisted type) then M^3 is of untwisted type (resp. of twisted type).*

(3) *The formal product $\times$ given in (1.1) is associative.*

The following proposition summarizes the graded dimensions of irreducible V_L^+-modules (cf. Section 10.5 of [FLM]).

Proposition 2.5. *Let L be an even lattice of rank n. Let $\lambda \in L^\circ \cap (L/2)$, $\mu \in L^\circ \setminus (L/2)$ and $\chi \in X(Z(\hat{L}/K_L))$ such that $\lambda \notin L$. Then we have*

$$\begin{aligned}
\dim_*[\mu] &= \frac{\Theta_{\mu+L}(q)}{\Phi(q)^n},\\
\dim_*[0]^{\pm} &= \frac{1}{2}\Big(\frac{\Theta_L(q)}{\Phi(q)^n} \pm \frac{\Phi(q)^n}{\Phi(q^2)^n}\Big),\\
\dim_*[\lambda]^{\pm} &= \frac{\Theta_{\lambda+L}(q)}{2\Phi(q)^n},\\
\dim_*[\chi]^{\pm} &= \frac{\dim T_\chi q^{\frac{n}{16}}}{2}\Big(\frac{\Phi(q)^n}{\Phi(q^{1/2})^n} \pm \frac{\Phi(q^2)^n\Phi(q^{1/2})^n}{\Phi(q)^{2n}}\Big),
\end{aligned}$$

where $\Phi(q) = \Pi_{i=1}^{\infty}(1-q^i)$ and $\Theta_U(q) = \sum_{\alpha\in U} q^{\langle\alpha,\alpha\rangle/2}$ for $U \subset \mathbb{R}^n$. In particular, first few coefficients are given as follows:

$$\begin{aligned}
\dim_*[0]^+ &\in 1 + \frac{|L_2|}{2}q + \Big(\frac{n|L_2|+|L_4|}{2} + \binom{n+1}{2}\Big)q^2 + q^3\mathbb{Z}[q],\\
\dim_*[0]^- &\in \Big(\frac{|L_2|}{2} + n\Big)q + \Big(\frac{n|L_2|+|L_4|}{2} + n\Big)q^2 + q^3\mathbb{Z}[q],\\
\dim_*[\lambda]^{\pm} &\in \frac{|(\lambda+L)_{\iota(\lambda+L)}|}{2}q^{\iota(\lambda+L)/2} + q^{\iota(\lambda+L)/2+1}\mathbb{Z}[q],\\
\dim_*[\chi]^+ &\in \dim T_\chi q^{\frac{n}{16}} + \binom{n+1}{2}\dim T_\chi q^{\frac{n}{16}+1} + q^{\frac{n}{16}+2}\mathbb{Z}[q],\\
\dim_*[\chi]^- &\in n\dim T_\chi q^{\frac{n+8}{16}} + q^{\frac{n+8}{16}+1}\mathbb{Z}[q],
\end{aligned}$$

where $\iota(\lambda+L) = \min\{\langle v,v\rangle \mid v \in \lambda+L\}$.

2.2 Automorphism group of V_L

In this subsection, we describe the automorphism group of V_L after [DN1]. In particular, we study the action of some automorphisms of V_L preserving V_L^+ on the set of isomorphism classes of irreducible V_L^+-modules.

Let L be an even lattice of rank n. Let $\mathrm{Aut}(\hat{L^\circ})$ denote the automorphism group of the group $\hat{L^\circ}$. For $g \in \mathrm{Aut}(\hat{L^\circ})$, let $\bar{g}$ denote the linear isomorphism of L° defined by $\bar{g}(\alpha) = \overline{g(e(\alpha))}$, $\alpha \in L^\circ$. Let $O(L^\circ)$ denote the group of automorphisms of L° preserving the inner product $\langle \cdot, \cdot \rangle$. Set $O(\hat{L^\circ}) = \{g \in \mathrm{Aut}(\hat{L^\circ}) |\ g\omega_q = \omega_q,\ \bar{g} \in O(L^\circ)\}$. We view an element $f \in \mathrm{Hom}(L^\circ, \mathbb{Z}_q)$ as the automorphism of $\hat{L^\circ}$ which sends $e(\alpha)$ to $\omega_q^{f(\alpha)} e(\alpha)$. Hence we obtain an embedding $\mathrm{Hom}(L, \mathbb{Z}_q) \subset O(\hat{L^\circ})$. Similarly, the group $O(\hat{L})$, a group homomorphism from $O(\hat{L})$ to $O(L)$ and an embedding $\mathrm{Hom}(L, \mathbb{Z}_2) \subset O(\hat{L})$ are defined. By Proposition 5.4.1 of [FLM], we have the following exact sequences:

$$1 \to \mathrm{Hom}(L^\circ, \mathbb{Z}_q) \hookrightarrow O(\hat{L^\circ}) \xrightarrow{-} O(L^\circ) \to 1, \tag{2.2}$$

$$1 \to \mathrm{Hom}(L, \mathbb{Z}_2) \hookrightarrow O(\hat{L}) \xrightarrow{-} O(L) \to 1. \tag{2.3}$$

Remark 2.6. If $\langle L, L\rangle \subset 2\mathbb{Z}$ then the exact sequence (2.3) is split.

For $\beta \in L^\circ/2L^\circ$, let $f_\beta : L \to \mathbb{Z}_2$ denote the element of $\mathrm{Hom}(L, \mathbb{Z}_2)$ given by

$$f_\beta : \gamma \mapsto \langle \beta, \gamma \rangle \mod 2. \tag{2.4}$$

It is easy to see that $\mathrm{Hom}(L, \mathbb{Z}_2) = \{f_\beta |\ \beta \in L^\circ/2L^\circ\}$.

Since $O(\hat{L})$ fixes κ_L, an element of $O(\hat{L})$ is extended to an automorphism of $\mathbb{C}\{L\}$. Then $g \in O(\hat{L})$ is also extended to a linear automorphism of V_L by setting

$$g(h_1(n_1) \cdots h_k(n_k) \otimes a) = \bar{g}(h_1)(n_1) \cdots \bar{g}(h_k)(n_k) \otimes g(a), \tag{2.5}$$

where $n_i \in \mathbb{Z}_{<0}$, $h_i \in \mathfrak{h}_L$ and $a \in \mathbb{C}\{L\}$. By [FLM, Corollary 10.4.8], this extension gives an automorphism of the VOA V_L. Thus we have an injective group homomorphism $O(\hat{L}) \hookrightarrow \mathrm{Aut}(V_L)$ of which the image is identified with $O(\hat{L})$. For $g \in O(\hat{L})$, we call g a *lift of* $\bar{g}$. By [DN1], the automorphism group $\mathrm{Aut}(V_L)$ of V_L is described as follows:

Theorem 2.7. [DN1, Theorem 2.1] *Let L be an even lattice. Then* $\mathrm{Aut}(V_L) = N(V_L)O(\hat{L})$, *where* $N(V_L) = \langle \exp(v_0) |\ v \in (V_L)_1\rangle$ *is a normal subgroup of* $\mathrm{Aut}(V_L)$. *Moreover,* $\mathrm{Aut}(V_L)/N(V_L)$ *is isomorphic to a quotient group of* $O(L)$.

In the previous section, the involution θ_{V_L} of V_L is given to define the VOA V_L^+. Clearly θ_{V_L} is a lift of the (-1)-isometry which sends $\alpha \in L$ to $-\alpha$. The centralizer of θ_{V_L} in $\mathrm{Aut}(V_L)$ plays an important role in this paper.

Proposition 2.8. *The centralizer* $C_{\mathrm{Aut}(V_L)}(\theta_{V_L})$ *of* θ_{V_L} *in* $\mathrm{Aut}(V_L)$ *is* $C_{N(V_L)}(\theta_{V_L})O(\hat{L})$. *In particular, if* $L_2 = \phi$ *then* $C_{\mathrm{Aut}(V_L)}(\theta_{V_L}) = O(\hat{L}) \cong (\mathbb{Z}_2)^n.O(L)$.

Proof. By Theorem 2.7, $\mathrm{Aut}(V_L) = N(V_L)O(\hat{L})$. Since the (-1)-isometry belongs to $Z(O(L))$ and θ_{V_L} commutes with $\mathrm{Hom}(L,\mathbb{Z}_2)$, we have $O(\hat{L}) \subset C_{\mathrm{Aut}(V_L)}(\theta_{V_L})$. Since $N(V_L)$ is normal, the first assertion follows.

Suppose $L_2 = \phi$. Then we have $(V_L)_1 = \mathfrak{h}_L(-1)$. It suffices to show that $C_{N(V_L)}(\theta_{V_L}) \subset O(\hat{L})$. Let $h(-1) \in \mathfrak{h}_L(-1)$. Then the exponential $\exp(h(-1)_0)$ sends $x \otimes a \in V_L$ to $\exp(\langle h, \bar{a}\rangle)x \otimes a$. Hence $\exp(h(-1)_0)$ commutes with θ_{V_L} if and only if $h \in \pi\sqrt{-1}L^\circ$. It is easy to see that $\exp(\pi\sqrt{-1}\beta(-1)_0) = f_\beta$ for $\beta \in L^\circ$. Thus $C_{N(V_L)}(\theta_{V_L}) = \mathrm{Hom}(L,\mathbb{Z}_2) \subset O(\hat{L})$. $\square$

By the definition of V_L^+ and the proposition above, the group $C_{\mathrm{Aut}(V_L)}(\theta_{V_L})$ acts on V_L^+. We describe the action of $O(\hat{L})$ on the set of all isomorphism classes of irreducible V_L^+-modules of untwisted type.

Proposition 2.9. *For $g \in O(\hat{L})$, we have*

$$\begin{aligned} [\mu]^g &= [\bar{g}(\mu)],\ \mu \in L^\circ \setminus (L/2), \\ \{[\lambda]^{\pm,g}\} &= \{[\bar{g}(\lambda)]^{\pm}\},\ \lambda \in L^\circ \cap (L/2), \\ [0]^{\pm,g} &= [0]^{\pm}, \end{aligned}$$

where we regard $\bar{g} \in O(L)$ as an automorphism of L°. Moreover, if $\bar{g} = 1$ then for $\lambda \in L^\circ \cap (L/2)$,

$$[\lambda]^{\pm,g} = \begin{cases} [\lambda]^{\pm} & \text{if } g(2\lambda) = 0, \\ [\lambda]^{\mp} & \text{if } g(2\lambda) = 1, \end{cases}$$

where we regard g as an element of $\mathrm{Hom}(L,\mathbb{Z}_2)$.

Proof. Let α be a vector in L°. Since $O(L) = O(L^\circ)$, there exists an element h of $O(\hat{L^\circ})$ such that $\bar{g} = \bar{h}$. Then h is extended to a linear automorphism of V_{L° mapping $V_{\alpha+L}$ to $V_{\bar{g}(\alpha)+L}$ as in (2.5). In particular, its restriction to V_L is an element of $\mathrm{Aut}(V_L)$. By Theorem 2.7, there exists an element $f \in N(V_L)$ such that $g = f \circ h_{|V_L} \in O(\hat{L})$. By Proposition 1.3 (3), we have $V_{\alpha+L}^f \cong V_{\alpha+L}$. Hence we obtain $V_{\alpha+L}^g \cong h(V_{\alpha+L}) = V_{\bar{g}(\alpha)+L}$. In particular, we have $[\mu]^g = [\bar{g}(\mu)]$ for $\mu \in L^\circ \setminus (L/2)$, $\{[\lambda]^{\pm,g}\} = \{[\bar{g}(\lambda)]^{\pm}\}$ for $\lambda \in L^\circ \cap (L/2)$ and $[0]^{\pm,g} = [0]^{\pm}$.

We assume that $\bar{g} = 1$. Then $g = f_\beta$ for some $\beta \in L^\circ$. Let λ be an element of $L^\circ \cap (L/2)$. Let $\psi_\beta : V_{\lambda+L} \to V_{\lambda+L}$ be the isomorphism of V_L-modules defined by $x \otimes e(\lambda+\gamma) \mapsto (\sqrt{-1})^{\langle\beta,2\lambda\rangle} x \otimes e(\lambda+\gamma)$, $\gamma \in L$. Then $Y(gv,z)\psi_\beta w = \psi_\beta Y(v,z)w$ for $v \in V_L$, $w \in V_{\lambda+L}$. It is easy to see that $\theta_{V_L}\psi_\beta = (-1)^{\langle\beta,2\lambda\rangle}\psi_\beta\theta_{V_L}$. Therefore we obtain this proposition. $\square$

The following lemma is useful in calculating the automorphism group of V_L^+.

Lemma 2.10. *Let N be a sublattice of $L^\circ \cap (L/2)$ such that $L \subset N$. Set*

$$G = \{g \in O(\hat{L}) |\ [\lambda]^{\varepsilon,g} = [\lambda]^{\varepsilon} \text{ for } {}^\forall\varepsilon \in \{\pm\},\ {}^\forall\lambda \in N\}.$$

Then the following sequence

$$1 \to \{f \in \mathrm{Hom}(L, \mathbb{Z}_2) |\ f(2N) = 0\} \hookrightarrow G \xrightarrow{-} \{g \in O(L) |\ g = 1 \text{ on } 2N/2L\} \to 1$$

is exact.

Proof. By the exact sequence (2.2), it suffices to determine both $\mathrm{Hom}(L, \mathbb{Z}_2) \cap G$ and $\overline{G}$. By Proposition 2.9, we have $\mathrm{Hom}(L, \mathbb{Z}_2) \cap G = \{f \in \mathrm{Hom}(L, \mathbb{Z}_2) |\ f(2W) = 0\}$. Let us determine $\overline{G}$. By Proposition 2.9, $\overline{G}$ acts trivially on $2N/2L$. Hence we have $\overline{G} \subseteq G_0 = \{g \in O(L) |\ g = 1 \text{ on } 2N/2L\}$. Conversely, let h be an element of G_0. Then there exists $g \in O(\hat{L})$ such that $\bar{g} = h$ by the exact sequence (2.2). Let $s : 2N/2L \to \mathbb{Z}_2$ be the map defined by $g(a) = (-1)^{s(\bar{a})}a$ for $a \in \widehat{2N}$. Then s is a homomorphism of groups. Since $2N/2L$ is a subspace of $L/2L$ over $\mathbb{F}_2$, s is extended to $s_0 \in \mathrm{Hom}(L, \mathbb{Z}_2)$ such that $s_0 = s$ on $2N/2L$. Hence we have an element $s_0 g \in G$ such that $\overline{s_0 g} = h$. Therefore we obtain $\overline{G} = G_0$. □

2.3 Extra automorphisms of V_L^+

In this subsection, we construct some automorphism of V_L^+ called an *extra automorphism* for an even lattice L obtained by Construction B along the line of [FLM]. The term "extra automorphism" means that it does not fix the isomorphism class of V_L^-. We will show in Proposition 3.10 that an automorphism of V_L^+ is extra if and only if it is not obtained as the restriction of an automorphism of V_L to V_L^+.

Let C be a doubly even code of length n and let $\{\alpha_i |\ i \in \Omega_n\}$ be an orthogonal basis of $\mathbb{R}^n$ of norm 2. Let $L_B(C)$ be the lattice obtained by Construction B from C associated with $\{\alpha_i |\ i \in \Omega_n\}$. Then $L_B(C)$ is an even lattice of rank n by Proposition 1.5 (1). Set $L_A(C) = L_B(C) + \mathbb{Z}\alpha_1$, and fix $a_k \in \hat{L}_A(C)$ for each $k \in \Omega_n$ such that $\bar{a}_k = \alpha_k$. Let σ be the operator of $V_{L_A(C)}$ defined by

$$\sigma \ = \ \prod_{k=1}^{n} \exp\left((1+\sqrt{-2})(a_k)_0\right) \exp\left(\sqrt{\frac{-1}{2}}(a_k^{-1})_0\right) \exp\left((-1+\sqrt{-2})(a_k)_0\right). \qquad (2.6)$$

Then σ is an automorphism of the VOA $V_{L_A(C)}$.

Remark 2.11. The automorphism σ depends on the choice of the orthogonal basis $\{\alpha_i |\ i \in \Omega_n\}$ of norm 2 which is used in the construction of the lattice $L_B(C)$.

The following proposition is a slight modification of [FLM, Proposition 12.2.5].

Proposition 2.12. *Let J be a set of generators of an even lattice L of rank n and let $\{h_k |\ k \in \Omega_n\}$ be a basis of $\mathfrak{h}_L$. Then the VOA V_L^+ is generated by $\{h_i(-1)h_j(-1),\ a^+ |\ i, j \in \Omega_n\ a \in \hat{J}\}$, where $a^+ = a + \theta_{V_L}(a)$.*

Let J be a set of generators of $L_B(C)$ given in Proposition 1.4. By the proposition above, $\tilde{J} = \{\alpha_i(-1)\alpha_j(-1),\ a^+ |\ i, j \in \Omega_n\ a \in \hat{J}\}$ is a set of generators of $V_{L_B(C)}^+$. The following proposition follows from [FLM, Theorem 11.2.1, Corollary 11.2.4 and Proposition 12.2.1].

Proposition 2.13. [FLM] *The automorphism σ preserves the set $\tilde{J}$. In particular σ is an automorphism of $V_{L_B(C)}^+$. Moreover the order of σ is two.*

The following proposition is a slight generalization of [FLM, Proposition 12.2.7].

Proposition 2.14. *The automorphism σ sends $V_{L_B(C)}^-$ and $V_{\alpha+L_B(C)}^-$ to $V_{\alpha_1+L_B(C)}^+$ and $V_{\alpha_1+L_B(C)}^-$ respectively. In particular, $[0]^{-,\sigma} = [\alpha_1]^+$ and $[\alpha_1]^{-,\sigma} = [\alpha_1]^-$.*

This proposition shows that σ is an extra automorphism of $V_{L_B(C)}^+$.

Remark 2.15. In [FLM], it is assumed that the all-one codeword belongs to C. However, this assumption is not necessary to obtain the extra automorphism σ of $V_{L_B(C)}^+$.

3 Automorphism group of V_L^+

In this section, we study the automorphism group $\mathrm{Aut}(V_L^+)$ of V_L^+ by using its action on isomorphism classes of irreducible V_L^+-modules, and give a method of determining $\mathrm{Aut}(V_L^+)$.

3.1 VOA fixed by a finite abelian group

In this subsection, we study the automorphism groups of a simple VOA and its subVOA obtained as the fixed-point subspace under an abelian automorphism group. We will apply the results of this section to the VOAs V_L and V_L^+ in Section 3.3.

Let V be a simple VOA and let D be a finite abelian subgroup of $\mathrm{Aut}(V)$. For $d \in \mathrm{Hom}(D,\mathbb{C}^\times)$ we set $V(d) = \{v \in V|\ b(v) = d(b)v\ \forall b \in D\}$. Since the field $\mathbb{C}$ is algebraically closed and D is finite abelian, we have $V = \oplus_{d\in\mathrm{Hom}(D,\mathbb{C}^\times)} V(d)$. Let 1_D denote the identity element of $\mathrm{Hom}(D,\mathbb{C}^\times)$. Then $V(1_D)$ is a subVOA of V and $V(d)$ is a $V(1_D)$-module. Clearly the restriction $Y_{d_1,d_2}(u,z)$ of $Y(u,z)$ to $V(d_2)$ for $u \in V(d_1)$ is an intertwining operator of type $\binom{V(d_1d_2)}{V(d_1)\ V(d_2)}$. In this paper, we will use the following properties of $V(d)$ studied in [DM1].

Theorem 3.1. [DM1] *Let V be a simple VOA and let D be a finite abelian subgroup of $\mathrm{Aut}(V)$. Then $\{V(d)|\ d \in \mathrm{Hom}(D,\mathbb{C}^\times)\}$ is a set of non-isomorphic irreducible $V(1_D)$-modules. In particular, $V(1_D)$ is a simple VOA.*

The following proposition obtained in [DM2] is crucial for our purpose.

Proposition 3.2. [DM2, Proposition 5.3] *Let $V = (V, Y, \mathbf{1}, \omega)$ be a simple VOA and let D be a finite abelian subgroup of $\mathrm{Aut}(V)$. Suppose that $V(d_1) \times V(d_2) = V(d_1d_2)$ for all $d_i \in \mathrm{Hom}(D,\mathbb{C}^\times)$. If $\tilde{V} = (V, \tilde{Y}, \mathbf{1}, \omega)$ is also a simple VOA with $Y(v,z) = \tilde{Y}(v,z)$ for all $v \in V(1_D)$ then there exists an isomorphism $\tau : \tilde{V} \to V$ of VOAs such that $\tau_{|V(1_D)}$ is the identity operator and τ preserves $V(d)$ for all $d \in \mathrm{Hom}(D,\mathbb{C}^\times)$.*

Clearly, the normalizer $N_{\mathrm{Aut}(V)}(D)$ of D in $\mathrm{Aut}(V)$ preserves $V(1_D)$ and $\{V(d)|\ d \in \mathrm{Hom}(D,\mathbb{C}^\times)\}$ as a set of $V(1_D)$-modules. We note that if $g \in N_{\mathrm{Aut}(V)}(D)$ satisfies $g(V(d_1)) = V(d_2)$ then $V(d_1)^{g_{|V(1_D)}} \cong V(d_2)$ as $V(1_D)$-modules. Consider the restriction homomorphism

$$\varphi_V : N_{\mathrm{Aut}(V)}(D) \to H,$$

where

$$H = \left\{ h \in \mathrm{Aut}(V(1_D)) \,\middle|\, \{V(d)^h |\ d \in \mathrm{Hom}(D,\mathbb{C}^\times)\} = \{V(d)|\ d \in \mathrm{Hom}(D,\mathbb{C}^\times)\} \right\}. \tag{3.1}$$

Theorem 3.3. *Let $V = (V, Y, \mathbf{1}, \omega)$ be a simple VOA and let D be a finite abelian subgroup of* $\mathrm{Aut}(V)$*. Suppose that $V(d_1) \times V(d_2) = V(d_1 d_2)$ for all $d_i \in \mathrm{Hom}(D,\mathbb{C}^\times)$. Then the restriction homomorphism φ_V is surjective and* $\mathrm{Ker}\ \varphi_V = D$.

Proof. First we show the surjectivity of φ_V. Let g be an element of H. For any $d_1 \in \mathrm{Hom}(D,\mathbb{C}^\times)$, there exists a unique element d_2 of $\mathrm{Hom}(D,\mathbb{C}^\times)$ such that $V(d_2) \cong V(d_1)^g$ as $V(1_D)$-modules by Proposition 3.1 (2), and let $\psi_{d_2} : V(d_2) \to V(d_1)^g$ be an isomorphism of $V(1_D)$-modules. Then

$$Y_{1_D,d_1}(gv,z)\psi_{d_2} = \psi_{d_2} Y_{1_D,d_2}(v,z) \tag{3.2}$$

for $v \in V(1_D)$. In particular, we may choose

$$\psi_{1_D} = g \tag{3.3}$$

since $V(1_D)^g \cong V(1_D)$ and $Y_{1_D,1_D}(gv,z)g = gY_{1_D,1_D}(v,z)$ for $v \in V(1_D)$. Then we have a linear automorphism $\psi = \oplus_{d\in \mathrm{Hom}(D,\mathbb{C}^\times)}\psi_d$ of V. Set $Y^g(v,z) = \psi^{-1}Y(\psi(v),z)\psi$ for $v \in V$. By (3.3), it is easy to see that $\tilde{V} = (V, Y^g, \mathbf{1}, \omega)$ has a VOA structure. Moreover, for $v \in V$

$$\psi Y^g(v,z) = \psi\psi^{-1}Y(\psi(v),z)\psi = Y(\psi(v),z)\psi.$$

Hence ψ is an isomorphism of VOAs from V to $\tilde{V}$. By (3.2) and (3.3), $Y^g(v,z) = Y(v,z)$ for $v \in V(1_D)$. Applying Proposition 3.2 to V and $\tilde{V}$, we obtain an isomorphism $\tau : \tilde{V} \to V$ such that $\tau_{|V(1_D)}$ is the identity operator and τ preserves $V(d)$ for all $d \in \mathrm{Hom}(D,\mathbb{C}^\times)$. Then $\tau \circ \psi$ is an automorphism of the VOA V such that its restriction to $V(1_D)$ is g and $\tau \circ \psi \in N_{\mathrm{Aut}(V)}(D)$. Therefore $\varphi_V(\tau\circ\psi) = g$, and φ_V is surjective.

Next we determine the kernel of φ_V. Let h be an element of $\mathrm{Ker}\ \varphi_V$. By Schur's lemma, h acts on $V(d)$ by a scalar $\lambda_d \in \mathbb{C}^\times$ for each $d \in \mathrm{Hom}(D,\mathbb{C}^\times)$. In particular, $\lambda_{1_D} = 1$. By the assumptions on the fusion rules, we have $\lambda_{d_1}\lambda_{d_2} = \lambda_{d_1 d_2}$ for $d_i \in \mathrm{Hom}(D,\mathbb{C}^\times)$. Hence $h \in D$. Clearly $D \subseteq \mathrm{Ker}\ \varphi_V$. Therefore we obtain $\mathrm{Ker}\ \varphi_V = D$. $\square$

3.2 V_L^+ for a 2-elementary totally even lattice

In this subsection, we study the irreducible V_L^+-modules and their fusion rules for a 2-elementary totally even lattice L. The results of this section will be used for the description of $\mathrm{Aut}(V_L^+)$ when L is isomorphic to $\sqrt{2}E_8$ and the Barnes-Wall lattice of rank 16 in Section 4.2 and 4.3.

Let L be a 2-elementary totally even lattice L of rank n, namely $2L^\circ \subset L$ and both L and $\sqrt{2}L^\circ$ are even. Then $L^\circ/L \cong \mathbb{Z}_2^m$ for some $m \in \mathbb{Z}$. We note that the determinant of L is equal to 2^m. By Theorem 2.2 and Remark 2.3, V_L^+ has exactly 2^{m+1} non-isomorphic irreducible modules $V_{\lambda+L}^{\pm}$ $(\lambda + L \in L^\circ/L)$ of untwisted type. Let us consider irreducible V_L^+-modules of twisted type. By Proposition 2.1, such a V_L^+-module is uniquely determined by an element of $X(Z(\hat{L}/K_L))$. Since $2L^\circ \subset L$, we have $Z(\hat{L}/K_L) = 2\hat{L}^\circ/K_L$. Since $\langle \alpha, \alpha\rangle \in 4\mathbb{Z}$ for all $\alpha \in 2L^\circ$, the group $2\hat{L}^\circ/K_L$ is an elementary abelian 2-group. We choose a complement G to $\langle \kappa_L K_L\rangle$ in $2\hat{L}^\circ/K_L$, which is a subgroup of $2\hat{L}^\circ/K_L$ such that $2\hat{L}^\circ/K_L = G\langle \kappa_L K_L\rangle$ and $G \cap \langle \kappa_L K_L\rangle = \{K_L\}$. Then the map $G \to 2L^\circ/2L$, $a \mapsto \bar{a}$ is an isomorphism of groups, so we identify G with $2L^\circ/2L$. Let $\chi \in X(2\hat{L}^\circ/K_L)$. Then there exists a unique coset $\lambda + L \in L^\circ/L$ such that $\chi(\beta) = (-1)^{\langle \lambda, \beta\rangle}$ for $\beta \in 2L^\circ/2L$. We denote such a character by χ_λ. We note that this notation depends on the choice of the complement. Hence V_L^+ has exactly 2^{m+1} non-isomorphic irreducible modules $V_L^{T_{\chi_\lambda},\pm}$ $(\lambda+L \in L^\circ/L)$ of twisted type. Therefore V_L^+ has exactly 2^{m+2} non-isomorphic irreducible modules when L is a 2-elementary totally even lattice with determinant 2^m.

Let S_L be the set of all isomorphism classes of irreducible V_L^+-modules. By the arguments above, $S_L = \{[\lambda]^\pm, [\chi_\lambda]^\pm \mid \lambda + L \in L^\circ/L\}$. Let us describe the fusion rules of V_L^+. The fusion rules in this case is clearer than those in general case. We regard $\{\pm\}$ as the group of order 2 generated by $-$. Let $\nu : L^\circ \to \{\pm\}$ be the map defined by $\nu(\alpha) = +$ if $\langle \alpha, \alpha\rangle \in 2\mathbb{Z}$, and $\nu(\alpha) = -$ if $\langle \alpha, \alpha\rangle \in 1 + 2\mathbb{Z}$. Applying the results of [ADL] to our case, we obtain the following proposition.

Proposition 3.4. *Let L be a 2-elementary totally even lattice. Then the fusion rules of V_L^+ are described as follows:*

$$\begin{aligned}
[\lambda_1]^\delta \times [\lambda_2]^\varepsilon &= [\lambda_1 + \lambda_2]^{\delta\varepsilon},\\
[\lambda_1]^\delta \times [\chi_{\lambda_2}]^\varepsilon &= [\chi_{\lambda_1+\lambda_2}]^{\delta\varepsilon\nu(\lambda_2)\nu(\lambda_1+\lambda_2)},\\
[\chi_{\lambda_1}]^\delta \times [\chi_{\lambda_2}]^\varepsilon &= [\lambda_1 + \lambda_2]^{\delta\varepsilon\nu(\lambda_1)\nu(\lambda_2)},
\end{aligned}$$

where $\delta, \varepsilon \in \{\pm\} \cong \mathbb{Z}_2$ and $\lambda_1, \lambda_2 \in L^\circ$.

Now, we view the formal product $\times$ as a binary operation on S_L. By Proposition 1.1 and 2.4 (3), this operation is associative and commutative. It is easy to see that $W \times W = [0]^+$ for all $W \in S_L$. Hence S_L has a structure of an elementary abelian 2-group. So the set S_L forms an $(m+2)$-dimensional vector space over $\mathbb{F}_2$ under the fusion rules.

Remark 3.5. Let L be an even lattice. If $2L^\circ \not\subset L$ then $[\mu] \times [\mu] = [0] + [2\mu]$ for $\mu \in L^\circ \setminus (L/2)$. If $\sqrt{2}(L^\circ \cap (L/2))$ is not even then $[\lambda]^+ \times [\lambda]^+ = [0]^-$ for $\lambda \in L^\circ \cap (L/2)$

with $\langle\lambda,\lambda\rangle \in 1/2+\mathbb{Z}$. Hence the set S_L forms a vector space over $\mathbb{F}_2$ under the fusion rules if and only if L is 2-elementary totally even.

Let us consider the action of $O(\hat{L})$ on S_L.

Lemma 3.6. *For $f_\beta \in \mathrm{Hom}(L,\mathbb{Z}_2)$, $\lambda \in L^\circ$ and $\varepsilon \in \{\pm\}$, we have*

$$[\chi_\lambda]^{\varepsilon,f_\beta} = [\chi_{\lambda+\beta}]^{\varepsilon}.$$

In particular, f_β fixes all isomorphism classes of irreducible V_L^+-modules of twisted type if and only if $\beta \in L/2L^\circ$.

Proof. We set $\chi_\lambda^{(\beta)}(a) = (-1)^{\langle\beta,\bar{a}\rangle}\chi_\lambda(a)$, $a \in \hat{L}$. By the definition of χ_λ, we have $\chi_\lambda^{(\beta)} = \chi_{\lambda+\beta}$. It is easy to see that the actions of $Z(\hat{L}/K_L)$ on $[\chi_\lambda]^{\varepsilon,f_\beta}$ and $[\chi_\lambda^{(\beta)}]^{\varepsilon}$ are the same, hence $[\chi_\lambda]^{\varepsilon,f_\beta} = [\chi^{(\beta)}]^{\varepsilon}$. Thus we obtain $[\chi_\lambda]^{\varepsilon,f_\beta} = [\chi_{\lambda+\beta}]^{\varepsilon}$. □

Lemma 3.7. *Let F be the group consisting of all elements of $O(\hat{L})/\langle\theta_{V_L}\rangle$ that fix all isomorphism classes of irreducible V_L^+-modules. Then the sequence*

$$1 \to \{f_\beta |\ \beta \in L/2L^\circ\} \hookrightarrow F \xrightarrow{-} \{g \in O(L)|\ g = 1 \text{ on } 2L^\circ/2L\} \to 1$$

is exact.

Proof. By Lemma 2.10 and Lemma 3.6, we have $F \cap \mathrm{Hom}(L,\mathbb{Z}_2) = \{f_\beta |\ \beta \in L/2L^\circ\}$ and $\bar{F} \subseteq \{g \in O(L)|\ g = 1 \text{ on } 2L^\circ/2L\}$. Let us show that this inclusion becomes equality. Let h_0 be an automorphism of L acting trivially on $2L^\circ/2L$ and let h be an element of $O(\hat{L})$ such that $\bar{h} = h_0$. Then there exists a map $s : 2L^\circ/2L \to \mathbb{Z}_2$ such that $h(a) = (-1)^{s(\bar{a})}a$ for $a \in 2\hat{L}^\circ$. It is easy to see that s is a group homomorphism and $s = f_\beta$ for some $\beta \in L^\circ/2L^\circ$. Hence $f_\beta h$ fixes all isomorphism classes of irreducible modules of twisted type. By Theorem 3.4, fh_β also fixes all isomorphism class of irreducible V_L^+-modules of untwisted type. Hence $fh_\beta \in F$ and $\overline{f_\beta h} = h_0$. Thus we obtain $\bar{F} = \{g \in O(L)|\ g = 1 \text{ on } 2L^\circ/2L\}$. □

We saw that the set S_L of all isomorphism classes of irreducible V_L^+-modules forms a vector space over $\mathbb{F}_2$ under the fusion rules. We will introduce a quadratic form on S_L preserved by the action of $\mathrm{Aut}(V_L^+)$. We are interested in the cases $n \in 8\mathbb{Z}$ since we will consider 2-elementary totally even lattices $\sqrt{2}E_8$ and Λ_{16} later. So, in the rest of this subsection, we assume $n \in 8\mathbb{Z}$. Then by Proposition 2.5, $\dim_* W$ belongs to either $\mathbb{Z}[q]$ or $q^{1/2}\mathbb{Z}[q]$ for each $W \in S_L$.

Let $q_L : S_L \to \mathbb{F}_2$ be the map defined by

$$q_L(W) = \begin{cases} 0 & \text{if } \dim_*(W) \in \mathbb{Z}[q], \\ 1 & \text{if } \dim_*(W) \in q^{1/2}\mathbb{Z}[q], \end{cases}$$

where $W \in S_L$. In the other word,

$$
\begin{aligned}
q_L([\lambda]^{\pm}) &= \langle \lambda, \lambda \rangle \mod 2, \\
q_L([\chi_\lambda]^{+}) &= \begin{cases} 1, & (n \in 8+16\mathbb{Z}), \\ 0, & (n \in 16\mathbb{Z}), \end{cases} \\
q_L([\chi_\lambda]^{-}) &= \begin{cases} 0, & (n \in 8+16\mathbb{Z}), \\ 1, & (n \in 16\mathbb{Z}), \end{cases}
\end{aligned}
$$

where $\lambda \in L^{\circ}$. By Proposition 1.3 (2), $q_L(W) = q_L(W^g)$ for $W \in S_L$, $g \in \mathrm{Aut}(V_L^+)$.

Theorem 3.8. *Let L be a 2-elementary totally even lattice whose rank is a multiple of 8. Then the map q_L is a quadratic form associated with a non-singular symplectic form on S_L such that $q_L(W) = q_L(W^g)$ for all $W \in S_L$, $g \in \mathrm{Aut}(V_L^+)$.*

Proof. Let us check that the form $(\cdot,\cdot)_L : S_L \times S_L \to \mathbb{F}_2$ defined by $(x,y)_L = q_L(x) + q_L(y) - q_L(x+y)$ is non-singular symplectic. Let $\pi : \{\pm\} \to \mathbb{F}_2$ be an isomorphism of groups such that $\pi(+) = 0$ and $\pi(-) = 1$. Let λ_1, λ_2 be vectors in L° and let δ, ε be elements of $\{\pm\}$. It is easy to see that

$$
\begin{aligned}
q_L([\lambda_1]^{\delta}) + q_L([\lambda_2]^{\varepsilon}) + q_L([\lambda_1+\lambda_2]^{\delta\varepsilon}) &= \langle \lambda_1, \lambda_1 \rangle + \langle \lambda_2, \lambda_2 \rangle + \langle \lambda_1+\lambda_2, \lambda_1+\lambda_2 \rangle \\
&= 2\langle \lambda_1, \lambda_2 \rangle, \\
q_L([\lambda_1]^{\delta}) + q_L([\chi_{\lambda_2}]^{\varepsilon}) + q_L([\chi_{\lambda_1+\lambda_2}]^{\delta\varepsilon\nu(\lambda_2)\nu(\lambda_1+\lambda_2)}) &= \langle \lambda_1, \lambda_1 \rangle + \pi(\varepsilon) + \pi(\delta\varepsilon) \\
&\quad + \langle \lambda_2, \lambda_2 \rangle + \langle \lambda_1+\lambda_2, \lambda_1+\lambda_2 \rangle \\
&= 2\langle \lambda_1, \lambda_2 \rangle + \pi(\delta), \\
q_L([\chi_{\lambda_1}]^{\delta}) + q_L([\chi_{\lambda_2}]^{\varepsilon}) + q_L([\lambda_1+\lambda_2]^{\delta\varepsilon\nu(\lambda_1)\nu(\lambda_2)}) &= \pi(\delta) + \pi(\varepsilon) + \langle \lambda_1+\lambda_2, \lambda_1+\lambda_2 \rangle \\
&= \pi(\delta\varepsilon) + \langle \lambda_1+\lambda_2, \lambda_1+\lambda_2 \rangle.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
([\lambda_1]^{\delta}, [\lambda_2]^{\varepsilon})_L &= 2\langle \lambda_1, \lambda_2 \rangle, \\
([\lambda_1]^{\delta}, [\chi_{\lambda_2}]^{\varepsilon})_L &= 2\langle \lambda_1, \lambda_2 \rangle + \pi(\delta), \\
([\chi_{\lambda_1}]^{\delta}, [\chi_{\lambda_2}]^{\varepsilon})_L &= \langle \lambda_1+\lambda_2, \lambda_1+\lambda_2 \rangle + \pi(\delta\varepsilon).
\end{aligned}
$$

We will show that $(\cdot,\cdot)_L$ coincides with the non-singular symplectic form $(\cdot,\cdot)$ defined by

$$([\lambda_1]^{+}, [\lambda_2]^{+}) = 2\langle \lambda_1, \lambda_2 \rangle,\ ([\lambda_1]^{+}, [\chi_0]^{\pm}) = 0,\ ([\chi_0]^{+}, [\chi_0]^{-}) = 1,\ ([\chi_0]^{\pm}, [\chi_0]^{\pm}) = 0.$$

It is easy to see that

$$
\begin{aligned}
([\lambda_1]^{\delta}, [\lambda_2]^{\varepsilon}) &= ([\lambda_1]^{+}, [\lambda_2]^{+}) = 2\langle \lambda_1, \lambda_2 \rangle, \\
([\lambda_1]^{\delta}, [\chi_{\lambda_2}]^{\varepsilon}) &= ([\lambda_1]^{\delta}, [\lambda_2]^{\varepsilon\nu(\lambda_2)}) + ([\lambda_1]^{\delta}, [\chi_0]^{+}) = 2\langle \lambda_1, \lambda_2 \rangle + \pi(\delta), \\
([\chi_{\lambda_1}]^{\delta}, [\chi_{\lambda_2}]^{\varepsilon}) &= ([\lambda_1]^{\delta\nu(\lambda_1)}, [\lambda_2]^{\varepsilon\nu(\lambda_2)}) + ([\lambda_1]^{\delta\nu(\lambda_1)}, [\chi_0]^{+}) \\
&\quad + ([\chi_0]^{+}, [\lambda_2]^{\varepsilon\nu(\lambda_2)}) + ([\chi_0]^{+}, [\chi_0]^{+}) \\
&= 2\langle \lambda_1, \lambda_2 \rangle + \pi(\delta\nu(\lambda_1)) + \pi(\varepsilon\nu(\lambda_2)) \\
&= \langle \lambda_1+\lambda_2, \lambda_1+\lambda_2 \rangle + \pi(\delta\varepsilon).
\end{aligned}
$$

Hence we have $(\cdot,\cdot)_L = (\cdot,\cdot)$. In particular, the form $(\cdot,\cdot)_L$ is non-singular symplectic. $\square$

We consider the type of the quadratic from q_L. By the definition of q_L, S_L decomposes into an orthogonal direct sum $S_L = S_0 \oplus \{[0]^{\pm}, [\chi_0]^{\pm}\}$ with respect to the symplectic form associated with q_L, where $S_0 = \{[\lambda]^+ |\ \lambda \in L^\circ/L\}$ is a subspace of S_L. On the other hand, L°/L is a vector space over $\mathbb{F}_2$ equipped with a non-singular quadratic form $q(\lambda+L) = \langle\lambda,\lambda\rangle \mod 2$, $\lambda \in L^\circ/L$. The linear isomorphism $f : L^\circ/L \to S_0$, $\lambda+L \mapsto [\lambda]^+$ show that the types of the restriction of q_L to S_0 and q are the same. Since the type of the restriction of q_L to $\{[0]^{\pm}, [\chi_0]^{\pm}\}$ is plus, the type of q_L is equal to that of q. Hence we obtain the following remark.

Remark 3.9. The type of q_L is equal to that of the quadratic form q on L°/L given by $q(\lambda + L) = \langle\lambda,\lambda\rangle \mod 2$, $\lambda \in L^\circ/L$.

3.3 Action of the automorphism group of V_L^+ on isomorphism classes of the irreducible modules

In this subsection, we study the action of $\mathrm{Aut}(V_L^+)$ on the set of isomorphism classes of irreducible V_L^+-modules. Among other things, we consider the stabilizer of the isomorphism class $[0]^-$ of V_L^- and the orbit of $[0]^-$.

Let L be an even lattice of rank n and let S_L be the set of all isomorphism classes of irreducible V_L^+-modules. Let H_L be the stabilizer of $[0]^-$ under the action of $\mathrm{Aut}(V_L^+)$ on S_L, that is, $H_L = \{g \in \mathrm{Aut}(V_L^+) |\ [0]^{-,g} = [0]^-\}$.

First we determine the group H_L. It is easy to see that V_L and $\langle\theta_{V_L}\rangle$ satisfy the assumption of Theorem 3.3. We note that $C_{\mathrm{Aut}(V_L)}(\theta_{V_L}) = N_{\mathrm{Aut}(V_L)}(\langle\theta_{V_L}\rangle)$ since the order of θ_{V_L} is two. By Theorem 3.3, the restriction homomorphism from $C_{\mathrm{Aut}(V_L)}(\theta_{V_L})$ to H_L is surjective and its kernel is $\langle\theta_{V_L}\rangle$. Hence we have the following proposition.

Proposition 3.10. *The stabilizer H_L of $[0]^-$ is $C_{\mathrm{Aut}(V_L)}(\theta_{V_L})/\langle\theta_{V_L}\rangle$.*

As an application of Proposition 3.10, we have the following proposition.

Proposition 3.11. *The group $\mathrm{Aut}(V_L^+)$ is finite if and only if L has no roots.*

Proof. Suppose that L has roots, and let α be a root of L. Then $v = e(\alpha) + \theta_{V_L}(e(\alpha))$ is a vector in $(V_L^+)_1$. It is easy to see that $v_0(e(2\alpha) + \theta_{V_L}(e(2\alpha))) \neq 0$, which implies that $\exp(v_0)$ generates an infinite subgroup of $\mathrm{Aut}(V_L^+)$.

Conversely, suppose that L has no roots. Then $C_{\mathrm{Aut}(V_L)}(\theta_{V_L})$ is finite by Proposition 2.8. By Theorem 2.2 and Remark 2.3, the cardinality of S_L is finite. Hence $|\mathrm{Aut}(V_L^+)| \leq |S_L| \cdot |C_{\mathrm{Aut}(V_L)}(\theta_{V_L})/\langle\theta_{V_L}\rangle| < \infty$. $\square$

Set $\mathcal{O} = \{W \in S_L |\ W \times W = [0]^+,\ \dim_* W = \dim_*[0]^-\}$. Then we have the following lemma.

Lemma 3.12. *Let L be an even lattice of rank n.*

(1) *If* $n \neq 8, 16$ *then* $\mathcal{O} \subseteq \{[0]^-, [\lambda]^{\pm} |\ \lambda \in L^\circ \cap (L/2),\ |(\lambda + L)_2| = 2n + |L_2|\}$.

(2) *If* $n = 8$ *then* $\mathcal{O} \subseteq \{[0]^-, [\lambda]^{\pm},\ [\chi]^- |\ \lambda \in L^\circ \cap (L/2),\ |(\lambda + L)_2| = 2n + |L_2|,\ \chi \in X(Z(\hat{L}/K_L))\}$.

(3) *If* $n = 16$ *then* $\mathcal{O} \subseteq \{[0]^-, [\lambda]^{\pm},\ [\chi]^+ |\ \lambda \in L^\circ \cap (L/2),\ |(\lambda + L)_2| = 2n + |L_2|,\ \chi \in X(Z(\hat{L}/K_L))\}$.

Proof. Proposition 2.4 (1) shows that $[\mu] \notin \mathcal{O}$ for $\mu \in L^\circ \setminus (L/2)$. Let λ be a vector in $L^\circ \cap (L/2)$ and let ε be an element of $\{\pm\}$. If $[\lambda]^\varepsilon \in \mathcal{O}$ then $\dim(V_{\lambda+L}^\varepsilon)_1 = \dim(V_L^-)_1$. By Proposition 2.5, we have $|(\lambda + L)_2| = 2n + |L_2|$. Let χ be an element of $X(Z(\hat{L}/K_L))$. By Proposition 2.5, if $[\chi]^+ \in \mathcal{O}$ then $n = 16$, and if $[\chi]^- \in \mathcal{O}$ then $n = 8$. $\square$

Next we consider the orbit Q_L of the isomorphism class $[0]^-$ of V_L^- or $Q_L = \{W \in S_L |\ W^g = [0]^-,\ ^\exists g \in \mathrm{Aut}(V_L^+)\}$. Let us study an even lattice L satisfying the following condition:

(I) Q_L contains isomorphism classes of irreducible V_L^+-modules of twisted type.

Lemma 3.13. *Let L be an even lattice satisfying (I). Then $2L^\circ \subset L$.*

Proof. Suppose that $2L^\circ \not\subset L$ and let $\mu \in L^\circ \setminus (L/2)$. By Proposition 2.4 (1), the fusion rule $[0]^- \times [\mu] = [\mu]$ holds. By the condition (I), there exists $g \in \mathrm{Aut}(V_L^+)$ such that $[0]^{-,g}$ is of twisted type. Then the fusion rule $[0]^{-,g} \times [\mu]^g = [\mu]^g$ contradicts Proposition 2.4 (2). $\square$

The identity relating the theta series of a lattice and the Dedekind η-function is crucial in the following proposition.

Proposition 3.14. *Let L be an even lattice satisfying (I). Then L is a 2-elementary totally even lattice of rank 8 or 16. Moreover, if L does not have roots then L is isomorphic to either $\sqrt{2}E_8$ or the Barnes-Wall lattice Λ_{16} of rank 16.*

Proof. By Lemma 3.12, the rank n of L is 8 or 16. By Lemma 3.13, $2L^\circ \subset L$. Let m be the integer given by $|L^\circ/L| = 2^m$. We note that 2^m is the determinant of L.

Let us check that $\langle v, v\rangle \in \mathbb{Z}$ for all $v \in L^\circ$. Set $\varepsilon = +$ if $n = 16$, and $\varepsilon = -$ if $n = 8$. Let χ be an element of $X(Z(\hat{L}/K_L))$ such that $[0]^{-,g} = [\chi]^\varepsilon$ for some $g \in \mathrm{Aut}(V_L^+)$. By Theorem 2.1, the dimension of T_χ is equal to $|L/(L \cap 2L^\circ)|^{1/2} = 2^{k/2}$, where $k = n - m$. By the equation $\dim_*[0]^- = \dim_*[\chi]^\varepsilon$ and Proposition 2.5, the theta series of L is described as follows:

$$n = 8:\ \Theta_L(q) = \frac{\eta(\tau)^{16}}{\eta(2\tau)^8} + 2^{\frac{k}{2}} \frac{\eta(\tau)^{16}}{\eta(\tau/2)^8} - 2^{\frac{k}{2}} \frac{\eta(2\tau)^8 \eta(\tau/2)^8}{\eta(\tau)^8}, \tag{3.4}$$

$$n = 16:\ \Theta_L(q) = \frac{\eta(\tau)^{32}}{\eta(2\tau)^{16}} + 2^{\frac{k}{2}} \frac{\eta(\tau)^{32}}{\eta(\tau/2)^{16}} + 2^{\frac{k}{2}} \frac{\eta(2\tau)^{16} \eta(\tau/2)^{16}}{\eta(\tau)^{16}}, \tag{3.5}$$

where $q = e^{2\pi\sqrt{-1}\tau}$ and $\eta(\tau) = q^{1/24}\Pi_{i=1}^{\infty}(1-q^i)$, which is called the Dedekind η-function. The following formulas are well-known:

$$\begin{aligned} \Theta_L\left(\frac{-1}{\tau}\right) &= \left(\frac{\tau}{\sqrt{-1}}\right)^{n/2} \frac{1}{\sqrt{|L^\circ/L|}}\Theta_{L^\circ}(\tau), \\ \eta\left(\frac{-1}{\tau}\right) &= \left(\frac{\tau}{\sqrt{-1}}\right)^{1/2} \eta(\tau). \end{aligned}$$

Thus, we obtain the theta series of L°:

$$n=8: \ \Theta_{L^\circ}(q) = \frac{\eta(\tau)^{16}}{\eta(2\tau)^8} + 2^{(16-k)/2}\frac{\eta(\tau)^{16}}{\eta(\tau/2)^8} - 2^4\frac{\eta(2\tau)^8\eta(\tau/2)^8}{\eta(\tau)^8}, \tag{3.6}$$

$$n=16: \ \Theta_{L^\circ}(q) = \frac{\eta(\tau)^{32}}{\eta(2\tau)^{16}} + 2^{(32-k)/2}\frac{\eta(\tau)^{32}}{\eta(\tau/2)^{16}} + 2^8\frac{\eta(2\tau)^{16}\eta(\tau/2)^{16}}{\eta(\tau)^{16}}. \tag{3.7}$$

In particular, $\Theta_{L^\circ}(q) \in \mathbb{Z}[q^{1/2}]$, which implies that $\langle v, v\rangle \in \mathbb{Z}$ for $v \in L^\circ$. Therefore L is 2-elementary totally even.

Suppose that L does not have roots. Then the coefficients of q in (3.4) and (3.5) are zero. So we have $k = 0$ if $n = 8$, and $k = 8$ if $n = 16$. Hence the determinant of L is 2^8. By Proposition 1.9, L is isomorphic to either $\sqrt{2}E_8$ or Λ_{16}. □

We will indeed check that $\sqrt{2}E_8$ and Λ_{16} satisfy the condition (I) later (see Proposition 4.7 and Remark 4.10).

In the rest of this section, we consider the orbit Q_L for an even lattice L of rank n satisfying the following condition:

(II) L has no roots and is isomorphic to neither $\sqrt{2}E_8$ nor Λ_{16}.

Then, by Proposition 1.3 (2) and (4), $Q_L \subseteq \mathcal{O}$. Hence, by Lemma 3.12 and the condition (II), we have

$$Q_L \subseteq \{[0]^-, [\lambda]^\pm |\ \lambda \in L^\circ \cap (L/2),\ |(\lambda + L)_2| = 2n\}. \tag{3.8}$$

We show that this inclusion becomes equality. The extra automorphisms given in Section 2.3 are crucial in the proof of the following theorem.

Theorem 3.15. *Let L be an even lattice of rank n satisfying (II).*

(1) $Q_L = \{[0]^-, [\lambda]^\pm |\ \lambda \in L^\circ \cap (L/2),\ |(\lambda + L)_2| = 2n\}$.

(2) *If $|Q_L| > 1$ then L is obtained by Construction B.*

Proof. Set $Q = \{[0]^-, [\lambda]^\pm |\ \lambda \in L^\circ \cap (L/2),\ |(\lambda + L)_2| = 2n\}$. By (3.8), we have $Q_L \subseteq Q$. If $Q = \{[0]^-\}$ then the assertion is trivial. So we assume $|Q| > 1$. Let $[\lambda]^+$ be an element of Q. By Proposition 1.8, the lattice L is obtained by Construction B associated with an orthogonal basis consisting of vectors in $(\lambda + L)_2$. By Proposition 2.6, there exists

$\sigma \in \mathrm{Aut}(V_L^+)$ such that $[0]^{-,\sigma} = [\lambda]^+$, hence $[\lambda]^+ \in Q_L$. By Proposition 2.9, there exists $f \in \mathrm{Hom}(L, \mathbb{Z}_2) \subset \mathrm{Aut}(V_L^+)$ such that $[\lambda]^{-,f} = [\lambda]^+$. Therefore $Q \subseteq Q_L$, and $Q = Q_L$. The arguments above show (2) as well. □

As an application of Theorem 3.15, we obtain the following proposition.

Proposition 3.16. *Let L be an even lattice without roots.*

(1) $\mathrm{Aut}(V_L^+)$ *is generated by $O(\hat{L})/\langle\theta_{V_L}\rangle$ and the extra automorphisms given in Section 2.3.*

(2) $O(\hat{L})/\langle\theta_{V_L}\rangle \subsetneq \mathrm{Aut}(V_L^+)$ *if and only if L is obtained by Construction B.*

Proof. First let us show (1). The stabilizer H_L of $[0]^-$ is isomorphic to $O(\hat{L})/\langle\theta_{V_L}\rangle$ by Proposition 2.8. Suppose that L is isomorphic to neither $\sqrt{2}E_8$ nor Λ_{16}. Let G be the subgroup of $\mathrm{Aut}(V_L^+)$ generated by H_L and the extra automorphisms given in Section 2.3. By the arguments in the proof of Theorem 3.15, G acts transitively on Q_L. Hence we obtain $G = \mathrm{Aut}(V_L^+)$. We will later check that this assertion holds when $L \cong \sqrt{2}E_8$ and Λ_{16} (see Remark 4.6 and 4.9).

Next we prove (2). If L is realized by Construction B, then V_L^+ has the extra automorphisms given by (2.6), hence $O(\hat{L})/\langle\theta_{V_L}\rangle \subsetneq \mathrm{Aut}(V_L^+)$. Conversely if $O(\hat{L})/\langle\theta_{V_L}\rangle \subsetneq \mathrm{Aut}(V_L^+)$ then $|Q| > 1$ since the stabilizer of $[0]^-$ is $O(\hat{L})/\langle\theta_{V_L}\rangle$ (cf. Proposition 2.8 and 3.10). By Theorem 3.15 (2), L is obtained by Construction B. □

In theorem 3.15, we determined Q_L. So we obtain a natural group homomorphism from $\mathrm{Aut}(V_L^+)$ to $Sym(Q_L)$, where $Sym(Q_L)$ is the group of permutations on Q_L. We denote this homomorphism by $\zeta_{V_L^+}$ The shape of $\mathrm{Aut}(V_L^+)$ can be described by the kernel and image of $\zeta_{V_L^+}$. Since $[0]^-$ belongs to Q_L, the kernel $\mathrm{Ker}\ \zeta_{V_L^+}$ of $\zeta_{V_L^+}$ is a subgroup of the stabilizer of $[0]^-$. By Theorem 2.7, Proposition 2.8 and 3.10, we can determine $\mathrm{Ker}\ \zeta_{V_L^+}$ in principle. However, $Sym(Q_L)$ is too large to describe the image $\mathrm{Im}\ \zeta_{V_L^+}$ of $\zeta_{V_L^+}$. So, we consider a certain structure on the orbit Q_L preserved by the action of $\mathrm{Aut}(V_L^+)$.

Proposition 3.17. *Let L be an even lattice of rank n satisfying (II). Set $P = Q_L \cup \{[0]^+\}$. Then for any $W^1, W^2 \in P$, there exists a unique element W^3 of P such that $W^1 \times W^2 = W^3$. Moreover P forms a vector space over $\mathbb{F}_2$ under this binary operation.*

Proof. If $Q_L = \{[0]^-\}$ then the assertion is trivial. We assume $|Q_L| \geq 2$. If $W^1 = [0]^+$ or $W^2 = [0]^+$ then the first assertion is trivial. Hence we may assume that $W^1, W^2 \in Q_L$. Then there exists $g \in \mathrm{Aut}(V_L^+)$ such that $W^{1,g} = [0]^-$. By Theorem 3.15, the set Q_L consists of isomorphism classes of irreducible modules of untwisted type. So $W^{2,g} = [\lambda]^\varepsilon$ for some $\lambda \in L^\circ \cap (L/2)$ and $\varepsilon \in \{\pm\}$. By Proposition 2.4 (1), the fusion rules $[0]^- \times [\lambda]^\pm = [\lambda]^\mp$ hold for all $\lambda \in L^\circ \cap (L/2)$. Thus we have $W^1 \times W^2 = g^{-1}(W^{1,g} \times W^{2,g}) = g^{-1}([0]^- \times [\lambda]^\varepsilon) = g^{-1}[\lambda]^{\varepsilon_0} \in P$, where $\{\varepsilon, \varepsilon_0\} = \{\pm\}$.

By Proposition 2.4 (3), the binary operation $\times$ is associative and commutative, hence this operation gives the structure of an abelian group on P. Since $\mathrm{Aut}(V_L^+)$ acts transitively on Q_L and $[0]^- \times [0]^- = [0]^+$, the order of any element of Q_L is 2. Therefore P is an

elementary abelian 2-group, namely P forms a vector space over $\mathbb{F}_2$ under the operation $\times$. $\square$

By Proposition 1.3 (4), $\mathrm{Aut}(V_L^+)$ preserves the group structure on P. We regard $\mathrm{Im}\ \zeta_{V_L^+}$ as a subgroup of the general linear group $GL(P)$ on the vector space P. Set

$$U_L = \left\{ \lambda + L \in (L^\circ \cap (L/2))/L \,\middle|\, |(\lambda + L)_2| = 2n \right\} \cup \{L\}. \tag{3.9}$$

Let $\lambda_1 + L, \lambda_2 + L$ be elements of U_L. By Proposition 2.4 (1), we have $[\lambda_1]^+ \times [\lambda_2]^+ = [\lambda_1 + \lambda_2]^\varepsilon$ for some $\varepsilon \in \{\pm\}$. It follows form Proposition 3.17 that $\lambda_1 + \lambda_2 \in U_L$. Hence the set U_L is a subspace of $(L^\circ \cap (L/2))/L$. Since $O(L) = O(L^\circ)$, the group $O(L)$ acts on U_L. Let ρ_L denote the natural group homomorphism from $O(L)$ to $GL(U_L)$. The image of $\zeta_{V_L^+}$ is described in the following proposition.

Proposition 3.18. *Let L be an even lattice satisfying (II) and let H_L be the stabilizer of the isomorphism class $[0]^-$ of V_L^-. Then $\zeta_{V_L^+}(H_L)$ is of shape $2^{\dim U_L}.\rho_L(O(L))$. Moreover if ρ_L is surjective then $\zeta_{V_L^+}$ is also surjective.*

Proof. The first assertion follows from Proposition 2.8 and 2.9. Suppose that ρ_L is surjective. If $\dim U_L = 0$ then $\dim P = 1$ and the second assertion is trivial. So we assume $\dim U_L \geq 1$. Clearly $\zeta_{V_L^+}(H_L)$ is a subgroup of the stabilizer G of $[0]^-$ under the action of $\mathrm{Im}\ \zeta_{V_L^+}$ on P. By the first assertion, $\zeta_{V_L^+}(H_L)$ is of shape $2^{\dim U_L}.GL(U_L)$. The one-point stabilizer under the natural action of $GL_{m+1}(2)$ on non-zero vectors in $\mathbb{F}_2^{m+1}$ is maximal and of shape $2^m.GL_m(2)$. Hence $\zeta_{V_L^+}(H_L) = G$. By Theorem 3.15, Proposition 3.16 (2) and the assumption that $\dim U_L \geq 1$, there exists $\sigma \in \mathrm{Aut}(V_L^+)$ such that $\zeta_{V_L^+}(\sigma) \notin G$. Hence we have $G \subsetneq \mathrm{Im}\ \zeta_{V_L^+} \subseteq GL(P)$. Therefore $\zeta_{V_L^+}$ is surjective. $\square$

3.4 Main results

We will summarize the results given in Section 3.2 and 3.3.

Let L be an even lattice of rank n and let Q_L be the orbit of the isomorphism class $[0]^-$ of the irreducible V_L^+-module V_L^- under the action of $\mathrm{Aut}(V_L^+)$ on the set S_L of all isomorphism classes of irreducible V_L^+-modules.

Theorem 3.19. (**Main result 1**)

(1) *The stabilizer H_L of $[0]^-$ is isomorphic to $C_{\mathrm{Aut}(V_L)}(\theta_{V_L})/\langle\theta_{V_L}\rangle$.*

(2) *If L has no roots then $C_{\mathrm{Aut}(V_L)}(\theta_{V_L})/\langle\theta_{V_L}\rangle = O(\hat{L})/\langle\theta_{V_L}\rangle \cong \mathrm{Hom}(L, \mathbb{Z}_2).(O(L)/\langle -1\rangle)$.*

Theorem 3.20. (**Main result 2**) *Suppose that L is a 2-elementary totally even lattice with determinant 2^m. Then S_L forms an $(m+2)$-dimensional vector space over $\mathbb{F}_2$ under the fusion rules. Moreover, if $n \in 8\mathbb{Z}$ then S_L has an $\mathrm{Aut}(V_L^+)$-invariant quadratic form associated with a non-singular symplectic form.*

Theorem 3.21. (**Main result 3**) *Suppose that Q_L contains isomorphism classes of irreducible V_L^+-modules of twisted type. Then L is 2-elementary totally even. Moreover if L has no roots then it is isomorphic to either $\sqrt{2}E_8$ or the Barnes-Wall lattice Λ_{16} of rank 16.*

Theorem 3.22. (**Main result 4**) *Suppose that L does not have roots and that L is isomorphic to neither $\sqrt{2}E_8$ nor Λ_{16}.*

(1) $Q_L = \{[0]^-, [\lambda]^\pm |\ \lambda \in L^\circ \cap (L/2),\ |(\lambda + L)_2| = 2n\}$

(2) *The set $P = Q_L \cup \{[0]^+\}$ forms a vector space over $\mathbb{F}_2$ under the fusion rules.*

See Section 3.2 for Theorem 3.20 and Theorem 3.21, and see Section 3.3 for Theorem 3.19 and 3.22.

The shape of $\mathrm{Aut}(V_L^+)$ for an even lattice L without roots is described in the following way. Suppose that L is isomorphic to neither $\sqrt{2}E_8$ nor Λ_{16}. By Theorem 3.22 (1), we obtain the orbit Q_L of $[0]^-$. Then by Theorem 3.22 (2), we have a natural group homomorphism $\zeta_{V_L^+}$ from $\mathrm{Aut}(V_L^+)$ to $GL(P)$. Clearly, its kernel is a subgroup of the stabilizer H_L of $[0]^-$. Hence by Theorem 2.7 and 3.19, we can determine the kernel of $\zeta_{V_L^+}$ in principle. The image $\mathrm{Im}\,\zeta_{V_L^+}$ of $\zeta_{V_L^+}$ is a subgroup of $GL(P)$ and the index of $\zeta_{V_L^+}(H_L)$ in $\mathrm{Im}\,\zeta_{V_L^+}$ is equal to $|Q_L|$. By Proposition 3.18 and Theorem 3.19, we determine $\zeta_{V_L^+}(H_L)$. By group theoretical facts on general linear groups, there are a few possibilities of the shape of $\mathrm{Im}\,\zeta_{V_L^+}$. Thus we can determine the image of $\zeta_{V_L^+}$, and obtain the shape of $\mathrm{Aut}(V_L^+)$.

When L is isomorphic to $\sqrt{2}E_8$ and Λ_{16}, the shape of $\mathrm{Aut}(V_L^+)$ will be determined by using Theorem 3.20 in Section 4.2 and 4.3.

4 Examples

In this section, as an application of the results of the previous section, we determine $\mathrm{Aut}(V_L^+)$ for many important lattices L.

4.1 Even unimodular lattices without roots

Let L be a positive-definite even unimodular lattice of rank n without roots. We note that if $n \le 16$ then L has roots (cf. [CS]). Hence $n \ge 24$. By Theorem 3.21 (2), the orbit of $[0]^-$ consists of only $[0]^-$. Hence the following theorem follows from Theorem 3.19.

Theorem 4.1. *Let L be an even unimodular lattice without roots. Then*

$$\mathrm{Aut}(V_L^+) \cong O(\hat{L})/\langle\theta_{V_L}\rangle \cong \mathrm{Hom}(L, \mathbb{Z}_2).(O(L)/\langle -1\rangle).$$

4.2 $\sqrt{2}R$ for an irreducible root lattice R of type ADE

Let R be an irreducible root lattice of type ADE. We note that the automorphism group of R was described in Table 1 and $O(R) = O(\sqrt{2}R)$. Set $L = \sqrt{2}R$. Then L has no roots. Let Q_L be the orbit of $[0]^-$. The following lemma is needed to describe $\mathrm{Aut}(V_L^+)$.

Lemma 4.2. (1)

$$|Q_L| = \begin{cases} 1 & \text{if } R = A_n\ (n \neq 3), E_6, E_7, \\ 3 & \text{if } R = A_3, D_n\ (n > 4), \\ 7 & \text{if } R = D_4. \end{cases}$$

(2) *Suppose that $|Q_L| > 1$ and that $R \neq E_8$. Let U_L be the subspace of $(L^\circ \cap (L/2))/L$ given in (3.9). Then the natural group homomorphism $\rho_L : O(L) \to GL(U_L)$ is surjective.*

Proof. The assertion (1) follows from Theorem 3.15. Clearly $|U_L| = (|Q_L| + 1)/2$. If $R \neq D_4$ then it follows from $\dim U_L \leq 1$ that ρ_L is surjective. The assertion (2) for the case $R = D_4$ is an easy exercise. □

The shape of $\mathrm{Aut}(V_{\sqrt{2}R}^+)$ except $R = E_8$ is described in the following theorem.

Theorem 4.3. *Let R be an irreducible root lattice not of type E_8 and set $L = \sqrt{2}R$. Then the shape of $\mathrm{Aut}(V_L^+)$ is described as follows:*

(i) *If $R = A_n$ $(n \neq 3)$, E_6 or E_7 then $\mathrm{Aut}(V_L^+) = O(\hat{L})/\langle \theta_{V_L} \rangle$.*

(ii) *If $R = A_3$ then $\mathrm{Aut}(V_L^+)$ is of shape $(2^2 : Sym_4).Sym_3$.*

(iii) *If $R = D_4$ then $\mathrm{Aut}(V_L^+)$ is of shape $(2^4 : Sym_4).GL_3(2)$.*

(iv) *If $R = D_n$ $(n \geq 5)$ then $\mathrm{Aut}(V_L^+)$ is of shape $(2^{2n-3} : Sym_n).Sym_3$.*

Proof. If $|Q_L| = 1$ then $\mathrm{Aut}(V_L^+) \cong O(\hat{L})/\langle \theta_{V_L} \rangle$ by Theorem 3.19. Hence (i) follows from Lemma 4.2 (1).

Suppose $|Q_L| > 1$. By Theorem 3.22 (2), the set $P = Q_L \cup \{[0]^+\}$ forms a vector space over $\mathbb{F}_2$ under the fusion rules. By Proposition 3.18 and Lemma 4.2 (2), the group homomorphism $\zeta_{V_L^+} : \mathrm{Aut}(V_L^+) \to GL(P)$ is surjective. Let us determine its kernel. By Remark 2.6, $O(\hat{L}) \cong \mathrm{Hom}(L, \mathbb{Z}_2) : O(L)$. Since $[0]^-$ belongs to Q_L, $\mathrm{Ker}\, \zeta_{V_L^+}$ is a subgroup of the stabilizer of $[0]^-$. By Theorem 3.19,

$$\mathrm{Ker}\, \zeta_{V_L^+} \subset O(\hat{L})/\langle \theta_{V_L} \rangle \cong \mathrm{Hom}(L, \mathbb{Z}_2) : (O(L)/\langle -1 \rangle). \tag{4.1}$$

By Proposition 3.18,

$$\zeta_{V_L^+}(O(\hat{L})/\langle \theta_{V_L} \rangle) \cong 2^{\dim P - 1} : GL_{\dim P - 1}(2). \tag{4.2}$$

By comparing (4.1) and (4.2), we obtain the shape of $\mathrm{Ker}\, \zeta_{V_L^+}$. Hence (ii), (iii) and (iv) follow. □

Note 4.4. The automorphism group of $V_{\sqrt{2}D_4}^+$ was determined in [MM]. In particular, $\mathrm{Aut}(V_{\sqrt{2}D_4}^+) \cong 2^6 : (GL_3(2) \times Sym_3)$.

The group $\mathrm{Aut}(V_{\sqrt{2}E_8}^+)$ was already obtained in [Gr2] (see [KM] for another proof) and the classification of 3-transposition groups is crucial in the papers. We will determine the automorphism group of $V_{\sqrt{2}E_8}^+$ without using the classification.

We recall the properties of the lattice $N = \sqrt{2}E_8$. By Example 1.6 (2), N is obtained by Construction B. By Proposition 1.9 (1), N is a 2-elementary totally even lattice of rank 8 with determinant 2^8. In particular, $N^\circ/N \cong 2^8$. It is easy to see that the type of the quadratic form $N^\circ/N \to \mathbb{F}_2$, $\lambda + N \mapsto \langle \lambda, \lambda\rangle \mod 2$ is plus (cf. [CS]). By Table 1, $O(N) \cong 2.O_8^+(2)$. By 2.3, $\mathrm{Aut}(\hat{N}) \cong 2^8.2.O_8^+(2)$.

Let S_N be the set of all isomorphism classes of irreducible V_N^+-modules. By Remark 3.9 and Theorem 3.20, S_N forms a 10-dimensional vector space over $\mathbb{F}_2$ under the fusion rules and has the $\mathrm{Aut}(V_N^+)$-invariant quadratic form q_N of plus type. Let $O(S_N)$ denote the orthogonal group on S associated with q_N and let $\xi_N : \mathrm{Aut}(V_N^+) \to O(S_N) \cong O_{10}^+(2)$ be the natural group homomorphism.

Let us determine both the kernel and image of ξ_N. It is well known that $O(N)/\langle -1\rangle \cong O_8^+(2)$ acts faithfully on N°/N. Since $N^\circ = N/2$, we have $\{h \in \mathrm{Hom}(N, \mathbb{Z}_2) |\ h(2N^\circ) = 0\} = \{1\}$. By Proposition 3.7, the kernel of ξ_N is trivial. Hence $\mathrm{Aut}(V_N^+) \subseteq O(S_N)$. By Proposition 2.6, there is an automorphism σ of V_N^+ such that $\sigma \notin O(\hat{N})/\langle \theta_{V_N}\rangle$. Hence we have $\langle O(\hat{N})/\langle\theta_{V_N}\rangle, \sigma\rangle \subseteq \mathrm{Aut}(V_N^+) \subseteq O(S_N)$. Since $O(\hat{N})/\langle\theta_{V_N}\rangle \cong 2^8 : O_8^+(2)$ is a maximal subgroup of $O_{10}^+(2)$ (cf. [ATLAS]), we have $\mathrm{Aut}(V_N^+) \cong O(S_N) \cong O_{10}^+(2)$.

Theorem 4.5. [Gr2, KM] *The automorphism group of $V_{\sqrt{2}E_8}^+$ is isomorphic to $O_{10}^+(2)$.*

Remark 4.6. The automorphism group of V_N^+ is generated by $O(\hat{N})/\langle\theta_{V_N}\rangle$ and the extra automorphisms given in Section 2.3, where $N = \sqrt{2}E_8$.

Now we study the orbit Q_N of $[0]^-$ more precisely. According to [ATLAS], the orbit decomposition of S_N under the action of the orthogonal group $O(S_N)$ is $2^{10} = 1+496+527$, where 1, 496, 527 mean the zero vector, the set of 496 non-isotropic vectors, the set of 527 non-zero isotropic vectors respectively. By the definition of the quadratic form q_N given in Section 3.2, $[0]^-$ is non-zero isotropic. Hence we obtain the following proposition.

Proposition 4.7. *The orbit Q_N is the set of all non-zero isotropic vectors in (S_N, q_N), namely*

$$Q_N = \{[0]^-, [\lambda]^\pm,\ [\chi]^-|\ \lambda \in N^\circ \cap (N/2),\ |(\lambda+N)_2| = 16,\ \chi \in X(Z(\hat{N}/K_N))\},$$

where $N = \sqrt{2}E_8$. In particular, the cardinality of Q_N is 527.

The results of this subsection are summarized in Table 2.

Table 2: Automorphism group of $V_{\sqrt{2}R}^{+}$

R	$\mathrm{Aut}(V_{\sqrt{2}R}^{+})$	$\lvert Q_{\sqrt{2}R}\rvert$
A_1	$\mathbb{Z}_2$	1
A_n $(n \neq 1,3)$	$2^n : Sym_{n+1}$	1
A_3	$(2^2 : Sym_4).Sym_3$	3
D_4	$(2^4 : Sym_4).GL_3(2)$	7
D_n $(n \geq 5)$	$(2^{2n-2} : Sym_n).Sym_3$	3
E_6	$2^6 : U_4(2) : 2$	1
E_7	$2^7 : Sp_6(2)$	1
E_8	$O_{10}^{+}(2)$	527

4.3 Barnes-Wall lattice of rank 16

We recall the properties of the Barnes-Wall lattice Λ_{16} of rank 16. By Example 1.6 (3), Λ_{16} is obtained by Construction B. By Proposition 1.9 (2), Λ_{16} is a 2-elementary totally even lattice of rank 16 with determinant 2^8 without roots. By Table 1, we have $O(\Lambda_{16}) \cong 2_+^{1+8} \cdot \Omega_8^+(2)$. It is easy to see that the type of the quadratic form $\Lambda_{16}^{\circ}/\Lambda_{16} \to \mathbb{F}_2$, $\lambda + \Lambda_{16} \mapsto \langle \lambda, \lambda \rangle \mod 2$ is plus (cf. [CS]).

Let $S_{\Lambda_{16}}$ be the set of all isomorphism classes of irreducible $V_{\Lambda_{16}}^{+}$-modules. By Remark 3.9 and Theorem 3.20, $S_{\Lambda_{16}}$ forms a 10-dimensional vector space over $\mathbb{F}_2$ under the fusion rules and has the $\mathrm{Aut}(V_{\Lambda_{16}}^{+})$-invariant quadratic form $q_{\Lambda_{16}}$ of plus type. Let $O(S_{\Lambda_{16}})$ denote the orthogonal group on S associated with $q_{\Lambda_{16}}$ and let $\xi_{\Lambda_{16}} : \mathrm{Aut}(V_{\Lambda_{16}}^{+}) \to O(S_{\Lambda_{16}}) \cong O_{10}^{+}(2)$ be the natural group homomorphism.

First we determine the kernel F of $\xi_{\Lambda_{16}}$. Let $H_{\Lambda_{16}}$ be the stabilizer of the isomorphism class $[0]^-$ of $V_{\Lambda_{16}}^{-}$. Then F is a normal subgroup of $H_{\Lambda_{16}}$. By Theorem 3.19, we have

$$H_{\Lambda_{16}} = C_{\mathrm{Aut}(V_{\Lambda_{16}})}(\theta_{V_{\Lambda_{16}}})/\langle \theta_{V_{\Lambda_{16}}} \rangle \cong \mathrm{Hom}(\Lambda_{16}, \mathbb{Z}_2).(O(\Lambda_{16})/\langle -1 \rangle). \tag{4.3}$$

In particular, $H_{\Lambda_{16}}$ is of shape $2^{16}.2^8.\Omega_8^+(2)$. We set $F_1 = F \cap \mathrm{Hom}(\Lambda_{16}, \mathbb{Z}_2)$. By Proposition 3.7, $F_1 \cong 2^8$ and $F/F_1 \cong 2^8$. In order to determine the structure of F, we use the automorphism $\sigma \in \mathrm{Aut}(V_{\Lambda_{16}}^{+})$ given in Section 2.3. It is easy to see that $\sigma^{-1} F_1 \sigma \neq F_1$ and the group F_1 is a subgroup of the center $Z(F)$ of F. Since $O(\hat{\Lambda}_{16})$ acts irreducibly on F/F_1 and σ does not normalize F_1, we have $F = Z(F)$. Since F_1 is an elementary abelian 2-group, F is generated by elements of order 2. Hence we obtain $F \cong 2^{16}$.

Next we determine the image of $\xi_{\Lambda_{16}}$. Set $G = H_{\Lambda_{16}}/F$. By (4.3) and the shape of F, the shape of G is $2^8.\Omega_8^+(2)$. Since $\sigma \notin H_{\Lambda_{16}}$, we have $\langle G, \sigma \rangle \subset O(S_{\Lambda_{16}})$. Since $2^8.\Omega_8^+(2)$ is a maximal subgroup of $\Omega_{10}^+(2)$ and $|O_{10}^+(2) : \Omega_{10}^+(2)| = 2$, the image of $\xi_{\Lambda_{16}}$ is of shape $\Omega_{10}^+(2)$ or $O_{10}^+(2)$.

Now we consider the action of $O(S_{\Lambda_{16}})$ on $S_{\Lambda_{16}}$. Similar arguments as in the previous section show that the cardinality of the orbit $Q_{\Lambda_{16}}$ of $[0]^-$ is less than or equal to 527,

which is the number of non-zero isotropic vectors in $S_{\Lambda_{16}}$. Hence we have

$$|\Omega_{10}^{+}(2)| \le |\mathrm{Im}\ \xi_{\Lambda_{16}}| \le 527 \cdot |G|.$$

Since $527 \cdot |G| = |\Omega_{10}^{+}(2)|$, we have $\mathrm{Im}\ \xi_{\Lambda_{16}} = \langle G, \sigma \rangle \cong \Omega_{10}^{+}(2)$. In particular, the cardinality of $Q_{\Lambda_{16}}$ is 527. Therefore we obtain an exact sequence

$$1 \to F \to \mathrm{Aut}(V_{\Lambda_{16}}^{+}) \to \langle G, \sigma \rangle \to 1. \tag{4.4}$$

Finally we show that this exact sequence is non-split. According to Theorem 1 in [Gr1], the shape of the group $B = O(\Lambda_{16})/\langle -1 \rangle$ is $2^8.\Omega_8^{+}(2)$ and it is non-split. Since $\Omega_8^{+}(2)$ is simple, B does not have a subgroup of shape $\Omega_8^{+}(2)$. Let A be the normal subgroup of B of shape 2^8. Set $K = \mathrm{Hom}(\Lambda_{16}, \mathbb{Z}_2) \subset H_{\Lambda_{16}}$. Then $H_{\Lambda_{16}}/K \cong B$. Suppose that (4.4) is split. Let C be a complement to F in $\mathrm{Aut}(V_{\Lambda_{16}}^{+})$. It is easy to see that $K(C \cap H_{\Lambda_{16}})/K \cong B/A$. This implies that $H_{\Lambda_{16}}/K$ has a subgroup of the shape $\Omega_8^{+}(2)$, which is a contradiction. Hence the exact sequence (4.4) is non-split.

Summarizing the results above, we have the following theorem.

Theorem 4.8. *The shape of the automorphism group of $V_{\Lambda_{16}}^{+}$ is $2^{16} \cdot \Omega_{10}^{+}(2)$.*

Remark 4.9. The automorphism group of $V_{\Lambda_{16}}^{+}$ is generated by $O(\hat{\Lambda}_{16})/\langle \theta_{V_{\Lambda_{16}}} \rangle$ and the extra automorphisms given in Section 2.3.

Remark 4.10. As in Proposition 4.7, the orbit $Q_{\Lambda_{16}}$ of $[0]^-$ is the set of all non-zero isotropic vectors in $(S_{\Lambda_{16}}, q_{\Lambda_{16}})$, namely

$$Q_{\Lambda_{16}} = \{[0]^-, [\lambda]^{\pm},\ [\chi]^{+} |\ \lambda \in \Lambda_{16}^{\circ} \cap (\Lambda_{16}/2),\ |(\lambda + \Lambda_{16})_2| = 32,\ \chi \in X(Z(\hat{\Lambda}_{16}/K_{\Lambda_{16}}))\}.$$

In particular the cardinality of $Q_{\Lambda_{16}}$ is 527.

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